Abstract

effective medium theories used to describe the overall properties of a material containing cracks on a length scale comparable to a wavelength are generally derived for an elastic medium. While adequate for describing properties of low matrix porosity materials such as carbonates, they provide a poor approximation once the matrix porosity has increased to an extent such that it plays a significant role in determining the matrix properties, such as with sandstones. Due to the significant difference in the behaviour of wave propagation in poroelastic media compared with that in elastic media, an alternative theory is required to adequately describe the full range of porosities encountered in crustal rock. Starting from the basic assumption that a saturated uncracked matrix can be described using Biot theory (Biot, 1962), we use the method of smoothing to develop an effective medium theory using the techniques used by Hudson (1980, 1994) for the equivalent elastic problem. The resulting theory may be used to describe the properties of a material containing a storage porosity associated with the background pore structure of the matrix and a transport porosity associated with the presence of ellipsoidal cracks or inclusions.

Biot Theory

The governing equations for the matrix material on the microscopic scale are those of elastodynamics in the solid components and linearised Navier-Stokes in the fluid, with continuity of velocity and normal traction at the boundary. We assume the process isothermal and homogeneous on the macroscopic scale and that homogenization reduces the equations to those of Biot on the macroscopic scale (Auriault, 1991). The governing equations may then be written as

\[ L \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} c_{ijkl} \partial_{i} u \partial_{j} + \rho \omega^{2} \delta \partial_{i} \\ -\delta \partial_{j} \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = 0, \]

(1)

in terms of the displacement \( u \) of the solid and the pressure \( p \) in the fluid, where \( c^{0} \) is the elastic stiffness tensor,

\[ \dot{\rho} = \rho + \rho_{j} \omega^{2} \theta, \]

(2)

\[ \dot{\alpha} = \alpha + \rho_{j} \omega^{2} \theta, \]

(3)

\[ \theta = -\left( m \rho_{j} \omega^{2} / \phi + i \omega \eta / K \right)^{-1}, \]

(4)

and
\[ \beta = (\alpha - \phi) / \kappa_s + \phi / \kappa_f, \]  
\hfill (5)  
in terms of the average density of the medium  
\[ \rho = (1 - \phi) \rho_s + \rho_f, \]  
\hfill (6)  
the effective stress coefficient  
\[ \alpha = 1 - \kappa / \kappa_s, \]  
\hfill (7)  
frequency \( \omega \), the density of the solid and fluid components \( \rho_s \) and \( \rho_f \), the tortuosity coefficient \( m \), porosity \( \phi \), dynamic viscosity \( \eta \), permeability \( K \) and the bulk moduli of the frame, grain and the fluid \( \kappa_s, \kappa_s \) and \( \kappa_f \). The symbol \( \delta_{ij} \) represents the Kronecker delta, \( \partial_i \equiv \partial / \partial x_i \) and \( \nabla^2 \) is the Laplacian operator.  

**Method of Smoothing**  
To determine the overall properties of a Biot medium permeated with cracks we use the method of smoothing (Keller, 1964) and assume that cracks exist on a scale length much larger than that of the microstructure (the pore space) but much less than that of a wavelength. Representing the total wave field during propagation in terms of a perturbation to the incident wave an averaging process yields an equation for the mean wave, which we may assume to approximate the observed wave under the aforementioned scaling conditions. The mean wave is found to obey a perturbation expansion in the parameter \( \epsilon \), indicating that the scattered wave is small, in some manner, in comparison with the incident wave:  
\[ \{ L - \epsilon \nu^* \int_V dV_s \bar{S}(x; \xi) + O(\epsilon^2) \} \begin{pmatrix} u \\ p \end{pmatrix} = 0, \]  
\hfill (8)  
in terms of the scattered wave \( \bar{S}(x, \xi) \) from an average inclusion centred at \( \xi \), where \( \nu^* \) is the number density of inclusions and the integral is over the total region of matrix and inclusions. To determine the scattered wave we use a representation theorem in terms of Green functions and the wave field inside an average inclusion. We approximate the strain in the inclusion due to the strain and pressure due to the incident field using the result of Berryman (1997), and approximate the displacement, pressure and pressure gradient in the inclusion with that due to the incident field by making what is essentially a Born approximation (Kuster and Toksöz, 1974). Thus, the governing equations of the uncracked isotropic matrix material (eq. 1) become those of an anisotropic equivalent medium given by  
\[ \begin{pmatrix} \epsilon^{**}_{ipq} \partial_p \partial_q + \hat{\rho}^* \omega^2 \delta_{ij} - \hat{\alpha}^*_{ip} \partial_p \\ -\hat{\alpha}^*_{jq} \partial_q \theta^* \nabla^2 - \beta^* \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = 0, \]  
\hfill (9)  
where  
\[ \hat{\rho}^* = \rho^* + \rho_f^* \omega^2 \theta^*, \]  
\hfill (10)  
\[ \hat{\alpha}^*_{ij} = \alpha^*_{ij} + \rho_f^* \omega^2 \theta^* \delta_{ij}, \]  
\hfill (11)  
and  
\[ \theta^* = \theta + s \theta^*, \]  
\hfill (12)  
where the effective density is given by  
\[ \rho^* = \rho + s \rho^* \]  
\hfill (13)  
and the effective poroelastic constants are given by
\[ c_{ipjq}^* = c_{ipjq}^0 + sc_{ipkl}^0 E_{klpq}, \]  
(14)  
\[ \alpha_{ij}^* = \alpha_{ij}^0 + s\alpha_{ij}^0 E_{klji} \]  
(15)  
and  
\[ \beta^* = \beta + s\beta^0 + s(\alpha^0)^2 \left(1 - \frac{E_{kkpp}}{3}\right) / \kappa^0. \]  
(16)

The parameters \( \rho^0, \theta^0, \) etc. are defined as the difference between the values of \( \rho, \theta, \) etc. inside the inclusion and those in the matrix, \( s \) is the volume density of the ellipsoidal inclusions and \( E \) is given in terms of the Eshelby (1957) tensor \( S \) by

\[ E = (Ss^0 + I)^{-1}, \]  
(17)  
where \( s^0 \), the elastic compliances of the matrix, is the inverse of \( c^0 \) and \( I \) is the fourth rank identity tensor.

**Some Results**

In addition to the three body waves equivalent to those observed in an elastic medium, a poroelastic medium also supports a slow compressional wave (a Biot wave). While this wave is too low to be observed in seismic data, its presence is nevertheless felt through the dissipation of energy. Unlike the body waves that are reduced in velocity due to the presence of inclusions, the speed of the Biot wave increases, as seen in Fig. 1, using typical matrix and inclusion properties, for vertical incidence on a set of vertically aligned cracks.

![Figure 1. The variation in Biot wave speed with the volume density of inclusions](image)

Due to the presence of the elliptical inclusions, the resulting effective medium exhibits orthorhombic symmetry and hence wave speeds will vary with offset and azimuth: the angles to the vertical and horizontal symmetry planes respectively. An example of this variation is provided in Fig. 2, where the two shear wave speeds are shown to vary with offset angle at a fixed azimuth inline with the narrowest dimension of the ellipsoids, using typical matrix and inclusion properties. A considerable degree of shear wave splitting is observed.

Changes in the porosity will also induce a change in the wave speeds; in addition to being explicitly present in the governing equations, a number of the material parameters change with porosity. The three fast wave velocities will all increase with an increase in porosity while the Biot wave velocity will decrease.
Discussion

Presenting the equations of poroelastodynamics in a manner comparable to those of elastodynamics has enabled the method of smoothing to be used to derive an effective medium theory for ellipsoidal cracks in an isotropic poroelastic matrix, the first such theory to include an explicit dependence on the orientation, size and content of the cracks. The theory has far reaching applications, such as providing considerably more accurate models of high porosity sandstone reservoirs in the hydrocarbon industry than provided by existing theories based on an elastic description.

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References


