Summary

We present new and improved equations for calculating the PS-wave (C-wave) conversion point in layered media with vertical transverse isotropy (VTI), or polar anisotropy. These equations are extensions of those in Thomsen (1999), but are derived from the Taylor-series expansion of the anisotropic ray equation, and are more accurate for large offsets. When the offset equals the reflector depth (offset-depth ratio $x/z = 1$), the conversion-point error is reduced from 5% of the offset to less than 0.5% (almost exact) for typical sedimentary rocks. For $x/z = 3$, the error is reduced from more than 10% to less than 1.5%. With the improved accuracy, the equations can help in C-wave data processing and parameter estimation.

Introduction

Determining the conversion point is an important step in P-SV converted-wave (C-wave) processing. Traditionally there are two approaches to this problem. One approach involves solving a fourth order polynomial equation, (e.g. Tessmer and Behle, 1988), and the other is based on the Taylor series expansion of the ray equation, including both asymptotic (first-order) and higher-order expansions (e.g. Thomsen, 1999). However, most of these equations have been derived for layered isotropic media.

Vertical transverse isotropy (VTI) is common in sedimentary rocks. Thomsen (1999) discussed various methods to account for the anisotropic effects using isotropic methods for conversion-point calculation, where the effects of anisotropy were only considered in the form of the effective velocity ratio $\gamma_{\text{eff}}$ ($\gamma_{\text{eff}}$ compensation). This approach may introduce significant conversion-point errors for a relatively small amount of anisotropy over modest offset ranges ($x/z = 1$) (Gaiser and Jackson 2000). For a homogeneous VTI medium, Li and Yuan (1999) derived a higher-order Taylor-series equation with sufficient accuracy up to $x/z = 3$. Here we extend the results of Li and Yuan (1999) to multi-layered media.

Generalized equations

Consider an $n$-layered VTI medium. Each layer is homogeneous with the following interval parameters for the $i$-th layer ($i=1,2,...,n$): $P$- and $S$-wave vertical velocities $V_{P0i}$ and $V_{S0i}$, vertical one-way travel times $\Delta t_{P0i}$ and $\Delta t_{S0i}$, respectively, and Thomsen (1986) parameters $\epsilon_i$ and $\delta_i$. The conversion-point offset $x_C$ for a C-wave ray converted at the bottom of the $n$-th layer and emerging at offset $x$ can be written as,

$$x_C = x \left( c_0 + \frac{c_2 x^2}{1 + c_3 x^2} \right)$$

where $c_3 = c_2 / (1 - c_0)$ (Thomsen 1999). Coefficients $c_0$ and $c_2$ are derived as (Yuan, 2000),

$$c_0 = \frac{\sum_{i=1}^{n} V_{P2i}^2 \Delta t_{P0i}}{\sum_{i=1}^{n} V_{P2i}^2 \Delta t_{P0i} + \sum_{i=1}^{n} V_{S2i}^2 \Delta t_{S0i}} \quad \text{and}$$

$$c_2 = \frac{\sum_{i=1}^{n} V_{P2i} \Delta t_{P0i}}{\sum_{i=1}^{n} V_{P2i} \Delta t_{P0i} + \sum_{i=1}^{n} V_{S2i} \Delta t_{S0i}}$$

(2)
\[
\begin{align*}
    c_2 &= \left[ \sum_{i=1}^{n} V_{S2i}^2 \Delta t_{S0i} \right] \left[ \sum_{i=1}^{n} V_{P2i}^4 \Delta t_{P0i} (1 + 8 \eta_i) \right] - \left[ \sum_{i=1}^{n} V_{P2i}^2 \Delta t_{P0i} \right] \left[ \sum_{i=1}^{n} V_{S2i}^4 \Delta t_{S0i} (1 - 8 \zeta_i) \right], \\
    &= 2 \left[ \sum_{i=1}^{n} V_{P2i}^2 \Delta t_{P0i} + \sum_{i=1}^{n} V_{S2i}^2 \Delta t_{S0i} \right],
\end{align*}
\]

where \( V_{P2i} \) and \( V_{S2i} \) denote the short-spread \( P \)- and \( S \)-wave NMO velocities, respectively, and
\[
\eta_i = (\epsilon_i - \delta_i) / (1 + 2\delta_i), \quad \zeta_i = (\gamma_{2i}^d / \gamma_{0i}^d) \eta_i = \gamma_{effi}^d \eta_i,
\]
\[
\gamma_{0i} = V_{P0i} / V_{S0i}, \quad \gamma_{2i} = V_{P2i} / V_{S2i}, \quad \text{and} \quad \gamma_{eff} = \gamma_{2i}^e / \gamma_{0i}.
\]

Note that \( \zeta_i \) is a new anisotropic parameter that determines the anisotropic effects on the \( S \)-wave leg of a \( C \)-wave ray, whereas \( \eta_i \) controls the effects on the corresponding \( P \)-wave leg.

Using Thomsen’s (1999) notation, we have
\[
\begin{align*}
    t_{P0} &= \sum_{i=1}^{n} \Delta t_{P0i}, \quad t_{S0} = \sum_{i=1}^{n} \Delta t_{S0i}, \quad t_{C0} = t_{P0} + t_{S0}, \\
    V_{P2}^2 &= \frac{1}{t_{P0}} \sum_{i=1}^{n} V_{P2i}^2 \Delta t_{P0i}, \quad V_{S2}^2 = \frac{1}{t_{S0}} \sum_{i=1}^{n} V_{S2i}^2 \Delta t_{S0i}, \quad t_{C0} V_{C2}^2 = t_{P0} V_{P2}^2 + t_{S0} V_{S2}^2, \\
    \gamma_0 &= \frac{t_{S0}}{t_{P0}}, \quad \gamma_2 = \frac{V_{P2}}{V_{S2}}, \quad \gamma_{eff} = \frac{\gamma_2^d}{\gamma_0}.
\end{align*}
\]

Introducing
\[
\eta_{eff} = \frac{1}{8t_{P0} V_{P2}^4} \left[ \sum_{i=1}^{n} V_{P2i}^4 \Delta t_{P0i} (1 + 8 \eta_i) - t_{P0} V_{P2}^4 \right]
\]
and
\[
\zeta_{eff} = \frac{-1}{8t_{S0} V_{S2}^4} \left[ \sum_{i=1}^{n} V_{S2i}^4 \Delta t_{S0i} (1 - 8 \zeta_i) - t_{S0} V_{S2}^4 \right],
\]
and substituting equations (7), (8) and (9) into equations (2) and (3), coefficients \( c_0 \) and \( c_2 \) become
\[
c_0 = \frac{\gamma_{eff}}{1 + \gamma_{eff}}, \quad \text{and} \quad c_2 = \frac{\gamma_{eff} (1 + \gamma_{eff})}{2t_{C0}^2 V_{C2}^4 (1 + \gamma_{eff})} \left[ \gamma_0 \gamma_{eff} - 1 \right] + 8(\eta_{eff} \gamma_0 \gamma_{eff} + \zeta_{eff}).
\]

All parameters have the same meanings as Thomsen (1999) except parameters \( \eta_{eff} \) and \( \zeta_{eff} \). \( \eta_{eff} \) was first introduced by Alkhalifah (1997), and \( \zeta_{eff} \) is a new parameter introduced here.

**Special cases**

**ISOTROPY.**- For layered isotropic media, parameters \( \eta_{eff} \) and \( \zeta_{eff} \) become
\[
\eta_{eff} = \frac{1}{8t_{P0} V_{P2}^4} \left[ \sum_{i=1}^{n} V_{P2i}^4 \Delta t_{P0i} - t_{P0} V_{P2}^4 \right] \quad \text{and} \quad \zeta_{eff} = \frac{-1}{8t_{S0} V_{S2}^4} \left[ \sum_{i=1}^{n} V_{S2i}^4 \Delta t_{S0i} - t_{S0} V_{S2}^4 \right],
\]
which control the residual layering effects on \( P \)- and \( S \)-waves, respectively. Ignoring these residual effects leads to,
\[
c_2 = \frac{\gamma_{eff} (1 + \gamma_{eff}) \left[ \gamma_0 (1 + \gamma_{eff}) - 1 \right]}{2t_{C0}^2 V_{C2}^4 (1 + \gamma_{eff})},
\]
which has the same form as the expressions given by Thomsen (1999).
SINGLE VTI LAYER.- For a single VTI layer, equation (9) reduces to that given by Li and Yuan (1999), since \( \eta_{\text{eff}} = \eta \), and \( \zeta_{\text{eff}} = \zeta \). Also noting that \( \zeta = \gamma^2 \eta \) in a single VTI layer, \( c_2 \) can be re-written as,

\[
c_2 = \frac{\gamma_{\text{eff}} (1 + \gamma_0)}{2 \varepsilon C_0 V_C^2 \gamma_0 (1 + \gamma_{\text{eff}})^2} \left[ (\gamma_0 \gamma_{\text{eff}} - 1) + 8 \eta \gamma_{\text{eff}} (\gamma_0 + \gamma_{\text{eff}}) \right].
\]

(12)

Accuracy

The accuracy of the equations is tested by synthetic modelling. Table 1 shows the parameters of a three-layer model. Each layer has a thickness of 500 metres. The conversion points calculated by the approximations are compared with the results of ray-tracing. Figure 1a shows the comparisons of the full equation (9) with ray tracing, and Figure 1b shows the comparison of equation (11) with ray tracing. For all three reflectors, the conversion point error is less than 0.5% for \( x/z = 1 \), and less than 1.5% for \( x/z = 3 \) (Figure 1a). However, if the anisotropy parameters \( \eta_{\text{eff}} \) and \( \zeta_{\text{eff}} \) are ignored, the error increases to 5% for \( x/z = 1 \), and to more than 10% for \( x/z = 3 \) (Figure 1b).

Implication for parameter estimation

With the improved accuracy, equations (1) and (9) may be used for estimating the effective velocity ratio and the anisotropic parameters if \( x_0 \), \( V_C \) and \( \gamma_0 \) are known. \( V_C \) can be determined from short-spread C-wave velocity analysis, and \( \gamma_0 \) from a coarse correlation of P- and C-wave stacked sections. At certain locations with distinct geological features, such as at a fault point, or the apex of an anticline, the conversion-point offset \( x_0 \) can be obtained by comparing the positive with the negative common-offset gathers. The effective velocity ratio \( \gamma_{\text{eff}} \) and the residual anisotropic term \( 8 \eta \gamma_{\text{eff}} (\gamma_0 + \gamma_{\text{eff}}) \) may then be recovered from equations (1) and (9) by inversion. Compared with C-wave moveout analysis (Yuan et al., 2001), this method does not require prior knowledge of the P-wave stacking velocity \( V_P \).

Discussion

For most earth models, one can expect that \( \eta_{\text{eff}} \) and \( \zeta_{\text{eff}} \) have the same sign. Thus the residual term \( 8 \eta \gamma_{\text{eff}} (\gamma_0 + \gamma_{\text{eff}}) \) in equation (9) may often be of the same order as the first term \( (\gamma_0 \gamma_{\text{eff}} - 1) \). This confirms Gaiser and Jackson’s (2000) numerical analysis that compensating for \( \gamma_{\text{eff}} \) only, i.e. ignoring the term \( 8 \eta \gamma_{\text{eff}} (\gamma_0 + \gamma_{\text{eff}}) \) in equation (9), is not sufficient to account for the anisotropic effects. They also found that an increase in \( \varepsilon \) or a decrease in \( \delta \) increases the errors in \( x_0 \). This can be easily explained by equation (9) or (12), since either an increase in \( \varepsilon \), or a decrease in \( \delta \) gives rise to an increase in \( \eta \) or \( \zeta \), and hence increases the errors in \( x_0 \).

It is also worth noting that the residual term after \( \gamma_{\text{eff}} \) compensation in equation (9) is determined by the anisotropic parameters \( \eta \) and \( \zeta \), instead of \( \delta \) and \( \sigma \). [As \( \delta \) does for P-wave, \( \sigma \) measures the relative difference between the S-wave stacking and vertical velocity, (Thomsen 1986)]. Yuan et al. (2001) find that the C-wave moveout signature is also determined by \( \eta \) and \( \zeta \).

Conclusions

Anisotropy has a significant effect on the conversion point and using \( \gamma_{\text{eff}} \) only is often insufficient to account for the effects, even when higher-order terms are considered. Based on a Taylor series expansion we have derived explicit expressions for the conversion point for layered VTI media. For small offsets, this new solution provides similar accuracy to Thomsen (1999), but it is much more accurate for large offsets. A new anisotropic parameter \( \zeta_{\text{eff}} \) is introduced to quantify the anisotropic effects on the S-wave leg of a C-wave ray. In a single VTI medium, \( \zeta_{\text{eff}} = \zeta = \gamma_{\text{eff}} \eta \). The new equations, with increased accuracy, may provide a reliable way to estimate anisotropy at locations with distinct geological features, such as, at a fault point.
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References


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<th>$\delta_i$</th>
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Table 1: A three-layer VTI model. The parameters are from Thomsen (1986).

Figure 1: Accuracy of $x_c$ approximations for the three-layer model in Table 1. (a) Results of equation (9) that considers both $\gamma_{\text{eff}}$ and the anisotropic residual term, and (b) results of equation (11) that only considers $\gamma_{\text{eff}}$ and ignores the residual anisotropic term. The relative error of $x_c$ is calculated as $(x_c^{(\text{exact})} - x_c^{(\text{approx})})/x$. 