Pure Quasi-P-wave Velocity-stress Equation in VTI Media
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SUMMARY
Simulation of multi-component anisotropic quasi-P-wavefield using full anisotropic elastic wave equation has high accuracy, however, the high computational cost and the difficulty of splitting elastic wavefield into de-coupled quasi-P- and quasi-S-wavefield limits its application. In this study, we derive a new one-order pure quasi-P-wave velocity-stress equation in VTI media. Our approach aims to projecting full anisotropic elastic wave equation in the direction of normalized P-wave polarization vector. Meanwhile, the Lowrank approximation method has been used to solve the proposed equation. We test the new one-order system with various models and the numerical examples demonstrate the proposed scheme is an efficient and stable method to simulate multi-component quasi-P-wavefield in VTI media.

INTRODUCTION
Anisotropy is common existed in sedimentary rock, which results anisotropic imaging and inversion playing a very important role in seismic prospecting. Although using full anisotropic elastic wave equation is the most accurate method to simulate anisotropic wavefield, its application is limited by the high computational cost and difficulty of splitting elastic wavefield into de-coupled P- and S-wavefield. One commonly used method is modeling quasi-P-wavefield with pseudo acoustic system (Alkhalifah, 2000) which is derivated by the assumption of setting the shear wave velocity along the symmetry axis to zero, \( V_{S0} = 0 \). However, the approximation acoustic wavefield calculated efficiently and kinematically accurately for reserve time migration (RTM) or full waveform inversion (FWI) is interfered with shear-wave and weak numerical instability (when \( \delta < \delta \)). To solve this problem, Liu et al. (2009) proposed a kind of pure quasi-P-wave equation and many researchers have studied different kinds of assumptions or numerical algorithms to calculate it. But most of the pulished works are focused on the scale wave equation while the vector acoustic data (Robertsson et al., 2008) have been drawing more and more attention.

In this paper, we build a new one-order pure quasi-P-wave equation in VTI media while the corresponding two-order pure quasi-P-wave displacement equation has been studied by Hou et al. (2014). Unlike the work of Liu et al. (2009) which has been derived from anisotropic velocity dispersion relation, the key of our method is projecting full anisotropic elastic wave equation in the direction of normalized P-wave polarization vector. Similar as the other proposed anisotropic pure mode wave equations, the new equation contains a space-wavenumber domain pseudo-differential operator that is difficult to be calculated by FD method directly. So we use the Lowrank approximation method (Fomel et al., 2013) to solve it. Our pure wave equation could output the anisotropic vector acoustic data which describes the kinematical property of displacement (or velocity) and stress correctly without share-wave noise. Meanwhile, benefits from the Lowrank method, the anisotropic P wavefield is free of numerical dispersion.

We test the new equation with various models. The numerical examples demonstrate the proposed scheme is an efficient and stable method to simulate multi-component quasi-P-wavefield in VTI media.

THEORY
Pure quasi-P-wave velocity-stress equation
In time/wavenumber domain, the one-order elastic wave displacement-stress equation of uniform infinite media can be expressed as:

\[
\rho \partial_t \mathbf{U} = \mathbf{L} \Sigma
\]

(1)

where \( \mathbf{L} \) represents the differential operator in time/wavenumber domain:

\[
\mathbf{L} = \begin{pmatrix}
  jk_x & 0 & 0 & jk_z & jk_y \\
  0 & jk_y & 0 & jk_z & 0 \\
  0 & jk_y & jk_y & jk_z & 0
\end{pmatrix},
\]

(2)

Here \( \Sigma = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}, \sigma_{z})^T \) and \( \mathbf{U} = (U_x, U_y, U_z)^T \) represent the stress tensor and displacement tensor in wavenumber domain, respectively; \( \rho \) and \( \mathbf{C} \) are the corresponding density and stiffness coefficient tensor.

According to the theory of Wavefield Vector Decomposition (Zhang and McMechan, 2010), the vector P wavefield can be decomposed from elastic wavefield exactly in the wavenumber domain:

\[
\mathbf{U}^P = \mathbf{A} \hat{\mathbf{U}},
\]

(3)

where

\[
\mathbf{A} = \begin{pmatrix}
  a_{xx} & a_{xy} & a_{xz} & a_{yx} & a_{yy} \\
  a_{yx} & a_{yy} & a_{yz} & a_{xy} & a_{zz} \\
  a_{xz} & a_{yz} & a_{zz} & a_{xz} & a_{yy} \\
  a_{yy} & a_{xy} & a_{yy} & a_{yx} & a_{zz} \\
  a_{zz} & a_{xy} & a_{zz} & a_{zz} & a_{yy}
\end{pmatrix},
\]

(4)

where \( (a_{xx}, a_{xy}, a_{xz})^T \) is normalized P-wave polarization vector, corresponding to the P-wave eigenvector of Christoffel equation (Carcione, 2007) and \( \hat{\mathbf{U}} = (\hat{U}_x^P, \hat{U}_y^P, \hat{U}_z^P)^T \) is P-wave displacement tensor in wavenumber domain.

Taking the Eq.3 into Eq.2, we can derive:

\[
\rho \partial_t \mathbf{U}^P = \partial_t (\mathbf{A} \hat{\mathbf{U}}) = \mathbf{AL} \hat{\Sigma}
\]

(5)

\[
\hat{\Sigma} = \mathbf{CL}^T \mathbf{A}^T \mathbf{A} \hat{\mathbf{U}} = \mathbf{CL}^T \mathbf{A}^T \mathbf{U}^P,
\]
where $A^T$ represents the generalized inverse of $A$. It’s can be proved that $A = A^T = A^{1/2}$, because the matrix of Eq.4 has only one eigenvalue which equals one. Therefore, we can define a operator in wavenumber domain: $T = AL$. And taking it into Eq.7:

$$\rho \partial_t \mathbf{V}^p = \Sigma$$
$$\Sigma = CT^T \mathbf{V}^p.$$  (6)

Eq.8 represents the pure quasi-P-wave displacement-stress equation in wavenumber domain. The corresponding velocity-stress equation is:

$$\rho \partial_t \mathbf{V}^p = \Sigma$$
$$\partial_t \Sigma = CT^T \mathbf{V}^p.$$  (7)

The core of this chapter is projecting full anisotropic elastic wave equation in the direction of normalized P-wave polarization vector. The similar projecting operator has been used by Cheng and Kang (2013). However, the difference is Cheng and Kang (2013) projected the elastic wave equation in the direction of propagation of elastic wave, $(k_x, k_y, k_z)^T$, which could not eliminate the shear-noise fully. So we use decomposition to solve the one-order system.

**Solving the proposed equation with lowrank method**

The proposed one-order system, Eq.6 or Eq.7, contains normalized P-wave polarization vector defined in the space-wavenumber domain. However, the solution is unavailable because of $N (N = \text{number of grids})$ times inverse FFT should be taken during one iteration. Due to the Lowrank decomposition (Fomel et al., 2013), the number of inverted Fourier transform could be decreased dramatically. We use decomposition to solve the one-order system.

In uniform infinite 2D-VTI media, according to Eq.7:

$$\rho \partial_t \mathbf{v}^p_x = jk_x p_x p_z \sigma_{xx} + jk_x p_x p_z \sigma_{zz} + jk_x p_x p_z \sigma_{xz} + jk_x p_x p_z \sigma_{zx},$$  (8)

According to differential operator splitting (Chu and Stoffa, 2012), Eq.8 can be decomposed to:

$$\rho \partial_t \mathbf{v}^p_x = \rho \partial_t \overline{\mathbf{v}}^p_x = \rho \partial_t (\mathbf{v}^p_{z1} + \mathbf{v}^p_{z2} + \mathbf{v}^p_{z3} + \mathbf{v}^p_{z4}),$$  (9)

where

$$\rho \partial_t \mathbf{v}^p_{z1} = jk_x p_x p_z \sigma_{xx},$$  (10)
$$\rho \partial_t \mathbf{v}^p_{z2} = jk_x p_x p_z \sigma_{zz},$$  (11)
$$\rho \partial_t \mathbf{v}^p_{z3} = jk_x p_x p_z \sigma_{xz},$$  (12)
$$\rho \partial_t \mathbf{v}^p_{z4} = jk_x p_x p_z \sigma_{zx}.$$  (13)

In the Eq.10, assuming that the $\mathbf{v}^p_{z1}$ and $\sigma_{xx}$ satisfy the plane wave solution in wavenumber-space domain:

$$\sigma_{xx}(t) = Qe^{jk \cdot x - j \omega t},$$  (14)

where $\omega = |k| \cdot v(x)$ is the circular frequency and $v(x)$ is the P wave phase velocity. So the solution of $\mathbf{v}^p_{z1}$ is:

$$\rho_{z1}^p(t + \Delta t) - \rho_{z1}^p(t) = jk_x p_x p_z e^{-j \omega t} - 1 \approx \frac{\mathbf{W}_L^{kxx} \cdot \mathbf{W}_R^{kxx}}{\delta \rho}.$$

According to Lowrank decomposition (Fomel et al., 2013):

$$k_x p_x p_z e^{-j \omega t} - 1 \approx \mathbf{W}_L^{kxx} \cdot \mathbf{W}_R^{kxx}.$$  (17)

Therefore, the solution of $\rho_{z1}^p(t + \Delta t)$ is:

$$\mathbf{V}^p(t + \Delta t) = \frac{1}{\rho} \mathbf{W}_L^{kxx} \cdot \mathbf{F}^{-1} \left\{ j \cdot \mathbf{W}_L^{kxx} \cdot \mathbf{F} \left\{ \sigma_{xx}(t) \right\} \right\}
+ \frac{1}{\rho} \mathbf{W}_L^{kzz} \cdot \mathbf{F}^{-1} \left\{ j \cdot \mathbf{W}_R^{kzz} \cdot \mathbf{F} \left\{ \sigma_{zz}(t) \right\} \right\}
+ \frac{1}{\rho} \mathbf{W}_L^{kxz} \cdot \mathbf{F}^{-1} \left\{ j \cdot \mathbf{W}_R^{kxz} \cdot \mathbf{F} \left\{ \sigma_{xz}(t) \right\} \right\}
+ \frac{1}{\rho} \mathbf{W}_L^{kzx} \cdot \mathbf{F}^{-1} \left\{ j \cdot \mathbf{W}_R^{kzx} \cdot \mathbf{F} \left\{ \sigma_{zx}(t) \right\} \right\}.$$  (18)

and solution of pure quasi-P-wave velocity-stress Eq.8 is:

$$\mathbf{V}^p(t + \Delta t) - \mathbf{V}^p(t) = \mathbf{W}_L^{kxx} \cdot \mathbf{F}^{-1} \left\{ j \cdot \mathbf{W}_R^{kxx} \cdot \mathbf{F} \left\{ \mathbf{V}(t + \Delta t) \right\} \right\},$$  (19)

$$\mathbf{V}^p(t + 1.5 \Delta t) - \mathbf{V}^p(t) = \mathbf{C} \cdot \mathbf{W}_L^{kxx} \cdot \mathbf{F}^{-1} \left\{ j \cdot \mathbf{W}_R^{kxx} \cdot \mathbf{F} \left\{ \mathbf{V}(t + \Delta t) \right\} \right\},$$  (20)

where $\mathbf{W}_L^{kxx}$ and $\mathbf{W}_R^{kxx}$ represents the Lowrank decomposition of wavenumber-space domain operator $\mathbf{Y}$; $\mathbf{W}_L^{kxx}$ and $\mathbf{W}_R^{kxx}$ represents the Lowrank decomposition of wavenumber-space domain operator $\mathbf{Y}_T^T$; $\mathbf{F}$ and $\mathbf{F}^{-1}$ represent the forward and backward Fourier transform, respectively, and $j$ is the imaginary unit.

**Computational flow**

In the above two section, we have already explained the basic idea of the pure quasi-P-wave velocity-stress equation in VTI media and the corresponding numerical algorithm. Now, we give the Computational flow.

For a given geology model, the main flow of modeling elastic wavefield by extrapolation equations (19) and (20) contains two steps: operators decomposition and wavefield extrapolation Fig. 1:

1. **Operators decomposition**, see Figure 1(a):
   - Calculate Christoffel equations to obtain P- and S-wave phase velocity and normalized P-wave polarization vector;
   - Calculate the Lowrank decomposition of extrapolation operators (19) and (20).

2. **Wavefield extrapolation**, see Figure 1(b):
   - Update $\mathbf{U}(t + \Delta t)$;
Anisotropic simulation

The computational flow of elastic-wave modeling is shown in Figure 1. (a): Operators decomposition; (b): Wave-field extrapolation.

EXAMPLES

In this section, we test the proposed method with two models: a simple homogenous model and a complex salt model.

2D homogeneous anisotropic model

The first model is a homogeneous anisotropic model. The computational grids are $301 \times 301$ points with the space interval 10m and the time interval 1ms. The corresponding anisotropic parameters are showed in the Table (1). We use a 25Hz Ricker wavelet source which is located at $(151, 151)$. Figure 2 shows the wavefield snapshots at 600ms and the anisotropic phase velocity.

Table 1: Elastic parameters (in Pa) for homogeneous model

<table>
<thead>
<tr>
<th>$C_{11}$</th>
<th>$C_{13}$</th>
<th>$C_{33}$</th>
<th>$C_{55}$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.20E + 6</td>
<td>4.40E + 6</td>
<td>4.00E + 6</td>
<td>1.44E + 6</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Obviously, the proposed numerical method can describe the kinematical property of anisotropic P-wave correctly and is free of share-wave noise even if the S-wave phase velocity doesn’t equal zero in this test. Meanwhile, the proposed method can output horizontal and vertical element simultaneously.

2D Hess VTI model

In the second example, we test the proposed method with Hess VTI model (left-top part) which contains multi-layer VTI medias and a high-velocity salt structure. The space interval in horizontal and vertical direction are 20m and 40m, respectively, and the time interval is 1ms. We also use a 25Hz Ricker wavelet source in this example where the results are showed in Figure 3. In this case, the analytical solution of Christoffel equation is

$$v^P(x) = \frac{1}{2}(a + \sqrt{a^2 - 4b})$$

and

$$A^P_x(x) = -\frac{c_1 k_x^2 + c_5 k_x^2 + 4(-a - \sqrt{a^2 - b})}{(c_1 c_5 + c_3) k_x k_z}$$

$$A^P_z(x) = 1$$

with

$$a = c_{11} k_x^2 + c_{55} k_x^2 + c_{33} k_z^2 + c_{55} k_z^2$$

$$b = c_{11} c_{55} k_x^2 - (c_{11} c_{33} + 2 c_{13} c_{55}) k_x^2 k_z^2 + c_{33} c_{55} k_z^4.$$ Benefits from the novel Lowrank decomposition, the proposed algorithm can output accurate anisotropic P-wavefield without numerical dispersion and share-wave interference, even if the model structure is complex and simulation parameters are large. Similar as the homogeneous model test, the proposed method could describe the kinematical property of anisotropic P-wave correctly and output horizontal and vertical wavefield simultaneously.

CONCLUSION

In this paper, a new pure quasi-P-wave equation is presented, combined with corresponding numerical modeling algorithm. This method is based on the wavefield vector decomposition and is accurate and applicable to simulating single mode P-wavefield correctly in complex anisotropic model without numerical dispersion and share-wave interference. Meanwhile, this forward methoding outputs horizontal and vertical wavefield simultaneously. Therefore, the proposed method can calculate the vector wavefield which can be used to process multi-component P-wave data. More research about increasing simulating speed and anisotropic imaging should be done in the future.

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Anisotropic simulation

Figure 2: Left: Snapshot of anisotropic P-wave \( (v_x) \) at 600ms; Middle: Snapshot of anisotropic P-wave \( (v_z) \) at 600ms; Right: Anisotropic P- and S-wave phase velocity.

Figure 3: Left: Part of anisotropic Hess model; Middle: Snapshot of anisotropic P-wave \( (v_x) \) at 750ms; Right: Snapshot of anisotropic P-wave \( (v_z) \) at 750ms.