Nonlinear PP and PS Joint Inversion based on the Zoeppritz Equations

Baoshan Song\textsuperscript{1,2,*}, Lixia Zhi\textsuperscript{1}, Shuangquan Chen\textsuperscript{1,2}, Lianbo Zeng\textsuperscript{1}, Xiang-Yang Li\textsuperscript{1,2,4}

1. State Key Laboratory of Petroleum Resources and Prospecting, China University of Petroleum (Beijing) 102249, People’s Republic of China
2. CNPC Key Laboratory of Geophysical Prospecting, China University of Petroleum (Beijing) 102249, People’s Republic of China
3. College of Science, China University of Petroleum (Beijing) 102249, People’s Republic of China
4. British Geological Survey, Murchison House, West Mains Road, Edinburgh EH9 3LA, UK

Summary
Density is an important parameter to distinguish lithology and estimate other petrophysical properties, such as porosity or fluid content. It is very difficult to estimate density reliably, even when using long-offset gathers in seismic data. Joint inversion of PP and PS data is a promising strategy for stabilizing and improving the results of inversion in estimating elastic parameters and density. This paper describes a strategy to explore nonlinear simultaneous PP and PS joint inversion based on the exact Zoeppritz equations. Thanks to the exact Zoeppritz equations, our joint inversion method is applicable to data sets with incident angles up to or beyond the critical angle, that is, usable for wide angle AVO inversion. To solve the highly nonlinear and ill-posed joint AVO inversion problem, the iterative regularizing Levenberg-Marquardt (IRLM) scheme is used, which can regularize the inversion problem within an algorithm that minimizes the misfit between the observed and the synthetic data at the same time. A comparison between joint inversion and PP-only inversion demonstrates that the joint inversion scheme based on the exact Zoeppritz equations can improve the quality of the subsurface parameter estimations, especially for the density term. The synthetic examples using a real field well-logging data and a Marmousi model also show that the proposed joint inversion is a reliable method, which is capable of accurately estimating the density parameter as well as P-wave velocity and S-wave velocity, even when the seismic data is noisy with signal-to-noise ratio (s/n) equal to 5.

Introduction
Knowledge of density and other elastic parameters such as the P-velocity and S-velocity of subsurface media is very important for characterizing conventional, unconventional and carbon dioxide (CO\textsubscript{2}) sequestered reservoirs. As one of the most important techniques used to invert for these subsurface physical parameters, amplitude versus offset (AVO), or angle (AVA) inversion has been frequently used in the modern petroleum industry. It is well known that the AVO or AVA inversion problem is ill-posed. That means that a little perturbation in the data results in a large perturbation in the estimation parameters. This is particularly true when one attempts to estimate the density term (Debski and Tarantola, 1995).

Combining PP and PS data in a joint multicomponent inversion scheme to estimate the elastic parameters and density has the potential to stabilize and improve the inversion results. A joint PP and PS inversion method was first presented by Stewart (1990), and Gary (2001) presented a practical method for the joint inversion of PP and PS reflection data and documented its performance. Veire and Landro (2006) discussed the singular value decomposition method for solving three-parameter joint PP and PS AVO inversion. Padhi and Mallic (2013) proposed a non-dominated sorting genetic algorithm to estimate density from the inversion of multicomponent prestack seismic waveform data. They all demonstrate that joint PP and PS AVO inversion is more stable and accurate. However, their inversions are all based on various approximations of the Zoeppritz equations.

Classical approximations of the Zoeppritz equations (Aki and Richards, 1980; Shuey, 1985) decouple the elastic parameters and facilitate intuitive understanding of how the parameter changes affect the reflection amplitudes (Russell et al., 2011). However in the near-surface where media have high velocity contrasts, and due to interference of coherent noise hindering the observation of reflections at short offsets, it is common to record reflections with angles of incidence above the critical angle, making various approximations of Zoeppritz equations difficult to use. On the other hand, most approximations of the Zoeppritz equations are functions of parameter contrasts, so that the inverted parameters based on these approximations are the different reflectivities of the parameters. To obtain the true values of the parameters, integration or additional steps must be performed, which will bring accumulation errors caused by indirect inversion. Here, we present a joint inversion method for estimating P-velocity, S-velocity and density directly based on the exact Zoeppritz equations. Because of the complexity in mathematical form of the Zoeppritz equations, the joint inversion problem is highly nonlinear and ill-posed. It is solved by using the IRLM (iteratively regularizing Levenberg-Marquardt) method (see our companion paper). Some schemes such as matrix...
factorization are also used to reduce the computational costs.

**Theory and Method**

Following weighted least-square estimation principles, we invert the unknown elastic parameters that are the P-velocity, S-velocity and density in this article, by minimizing the misfit errors between the observed seismic data and the modeled or synthetic data. Thus the joint inversion problem reduces to find \( m \) that minimizes the cost function

\[
F(m) = \sum_{i=1}^{p} w_i \left\| s_{pp}(\theta_i) - W_{pp}(\theta_i) \right\|^2 + \sum_{i=1}^{q} w_{i+p} \left\| s_{ps}(\varphi_i) - W_{ps}(\varphi_i) \right\|^2
\]

(1)

where, \( m = [V_{p1}, V_{p2}, \ldots, V_{pN}, V_{s1}, V_{s2}, \ldots, \rho_1, \rho_2, \ldots, \rho_3]^{T} \) are the elastic parameters to be inverted. \( s_{pp}(\theta) \) and \( s_{ps}(\theta) \) are the observed PP and PS seismic traces. \( R_{pp}(m, \theta) \) and \( R_{ps}(m, \theta) \) are the strata reflection coefficients corresponding to the incidence angle \( \theta \); \( W_{pp}(\theta) \) and \( W_{ps}(\theta) \) are the corresponding wavelet convolution matrices. The summation is done over all traces for the PP data \( (i=1,2,\ldots,p) \) and PS data \( (i=1,2,\ldots,q) \). The weight factor \( 0 \leq w_i \leq 1 \) can be used to reflect the varying quality of the two data sets for different angle gathers. In the synthetic test presented in this paper, the weight factor is set to 0.5, i.e. the two data sets are assumed to have equal quality. Equation (1) gives us a general expression for the simultaneous inversion of \( p \) PP angle stack and \( q \) PS stacks. Note that we can use a different wavelet for each of the PS angle stacks, as was done for each of the PP angle stacks.

In the forward modeling, the Zoeppritz equations (Aki and Richards, 1980) can be written in matrix form as

\[
AR = B,
\]

(2)

where,

\[
A = \begin{bmatrix}
\sin \alpha & \cos \beta & -\sin \alpha & \cos \beta \\
\cos \alpha & -\sin \beta & \cos \alpha & \sin \beta \\
\cos 2\beta & -V_{s1} \sin 2\beta & -\rho_1 V_{s1} \cos 2\beta & -\rho_1 V_{s1} \sin 2\beta \\
\sin 2\alpha & V_{s1} \cos 2\beta & \rho_1 V_{s1} \sin 2\alpha & \rho_1 V_{s1} \cos 2\beta
\end{bmatrix}
\]

\[
R = [R_{pp} \ R_{ps} \ T_{pp} \ T_{ps}]^{T}
\]

and \( V_{p1}, V_{p2}, V_{s1}, V_{s2}, \rho_1, \rho_2 \) are the P-wave velocity, S-wave velocity and density across an interface, \( \alpha, \beta \) are the P-wave angle of incidence and transmission across the interface, and \( \beta, \beta' \) are S-wave angle of incidence and transmission across the interface. \( R_{pp} \) and \( R_{ps} \) are PP and PS reflection coefficients respectively, \( T_{pp} \) and \( T_{ps} \) are PP and PS transmission coefficients, respectively. Snell’s law holds, as below:

\[
\frac{\sin \alpha}{V_{p1}} = \frac{\sin \alpha'}{V_{p2}} = \frac{\sin \beta}{V_{s1}} = \frac{\sin \beta'}{V_{s2}}.
\]

(3)

To obtain the Fréchet derivatives of reflection coefficients as parameters, we can simply differentiate equation (2) on both sides by \( x_i \), where \( x_i \) representing any parameter, giving

\[
A \frac{\partial R}{\partial x_j} = \frac{\partial B}{\partial x_j} - \frac{\partial A}{\partial x_j} R.
\]

(4)

Solving matrix equations (2) and (4) will give Fréchet derivatives of reflection coefficients as parameters.

Now the joint prestack inversion problem reduces to a nonlinear optimization problem, which in theory can be solved using the standard optimization solver. However, we should note that the goal of AVA inversion is far beyond just finding a ‘best-fit’ model. The observed seismic data are often noisy, and there may not be sufficient data. For these reasons, AVA inversion problems have non-unique solutions and are usually ill-posed. Some regularization skills must be used to stabilize the inversion problem. We use the IRLM scheme, which can regularize the inversion problem within an algorithm that also provides an approximation to a minimizer of equation (1), to solve the joint inversion problem. For a detailed introduction of this IRLM scheme, please see our companion paper.
Nonlinear PP and PS Joint Inversion based on the Zoeppritz Equations

Including PS data in the joint inversion would seem to increase the computational complexity. This is not true thanks to the matrix form of the Zoeppritz equations (2) and its partial derivative matrix (4), because we can get the PP and PS coefficients at the same time by solving matrix equations (2), as for the computation of partial derivatives. To get the Jacobian matrix, seven linear equations need to be solved for one layer, which is too much trouble. Luckily we can see that the seven linear equations are only different in the right side of the equation. So we can first do LU factorization on A, then we just solve seven triangle equations, which can be done by forward and back substitution. As we know, the amount of computation for the Gaussian elimination method is $2n^3/3$, while the amount of computation for forward and back substitution is only $n^2$, where $n$ represents the size of the matrix A.

Numerical examples

Firstly, the inversion method is tested on real field well logging data, shown in Figure 1. The time ranges from 1508ms to 2420ms with sampling interval of 2ms. The synthetic data is generated using a convolution model with the Zoeppritz equations and a zero-phase Ricker wavelet that is 35Hz dominant frequency. The range of incidence angles are from 10 to 45° degrees. Considering that seismic data are often contaminated with noise, we add Gaussian noise to the synthetic PP (middle) and PS (right) data with signal-to-noise ratio (s/n) equal to 5.

Figure 1. Well-log data, with red, green and blue lines representing P- and S-wave velocities and density, respectively, and synthetic angle-gathers based on Zoeppritz equations with signal to noise equal to 5.

Figure 2 illustrates the inversion results by PP and PS joint inversions (Figure 2a) and PP-only inversions (Figure 2b), based on the exact Zoeppritz equations. Both joint and PP-only inversions are solved using the IRLM scheme. As shown in Figure 2, the P-velocity and S-velocity inverted using both inversions coincide with the true well log model perfectly because we use the exact Zoeppritz equations in forward modeling. Notably, the quality of inverted density is enhanced greatly due to including PS data in the joint inversion. As the red arrow in Figure 2 shows, the density inverted using PP-only inversion in the time window about 1.6m-1.7m is different to the true model, while the result using joint inversion is almost the same as the true model.

Figure 2. Comparison of the inversion results: (a) the inverted P-, S-waves velocity and density curves using the joint PP and PS inversion; (b) the inverted results only using the PP data. The red arrows indicate some details which are enhanced in density parameter estimation.
Nonlinear PP and PS Joint Inversion based on the Zoeppritz Equations

In order to test the validity of our joint inversion method on a two-dimensional model, a part of a Marmousi model is chosen to build the test model. Figures 3a, 4a and 5a show the P-velocity, S-velocity and density of the test model in turn. As before, the synthetic seismic data is generated by convolving reflection coefficients with a Ricker wavelet. We also add Gaussian noise to the synthetic gathers with s/n=5 to test the stability of our method against noise. Figures 3b, 4b and 5b (P-velocity, S-velocity and density, respectively) show the inversion results of our PP and PS joint inversion based on the exact Zoeppritz equation. It is well known that the parameter changes of the Marmousi model are very large, so it is seldom used for testing inversion methods in literature. However, as is shown in Figures 3b, 4b and 5b, the inversion works quite well because the edges of the model in the P-wave velocity, S-wave velocity and density model are very clear. This once again proves that our PP and PS joint inversion based on the exact Zoeppritz equation is quite effective and reliable.

![Figure 3. Comparison of the original (a) and inverted (b) section of P-wave velocity (Vp) parameter, in a zoomed-in part of the Marmousi model](image)

![Figure 4. Comparison of the original (a) and inverted (b) section of S-wave velocity (Vs) parameter, in a zoomed-in part of the Marmousi model.](image)

**Figure 5.** Comparison of the original (a) and inverted (b) section of density parameter, in a zoomed-in part of the Marmousi model.

Conclusions

We develop and demonstrate a nonlinear PP and PS joint inversion based on the Zoeppritz equations, which is solved by using our IRLM scheme. To reduce the computation complexity, strategies such as matrix factorization are used. In order to test the validity of our joint inversion method, we applied the method to synthetic data. As is shown in the last section, this joint inversion method was successfully demonstrated with synthetic data. P-wave velocity, S-wave velocity and density were successfully estimated from noisy datasets with s/n=5. From comparison of inversion results between joint inversion and PP-only inversion, we can conclude that joint inversion based on the Zoeppritz equations can improve the quality of estimation of subsurface parameters, especially the estimation of density. Notably, our inversion method has a wider range of application, thanks to the exact Zoeppritz equations. For target areas with very large parameter contrasts such as northeastern Alberta, where the critical angle may be as low as 25 degrees (Downton, 2005), our method provides a reliable and promising method for estimating elastic parameters (especially density). To sum up, all our evidence demonstrates that the proposed joint inversion is a reliable method capable of accurately estimating the density parameter as well as P-wave velocity and S-wave velocity.

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