Boundary element simulation of multiple scattering of seismic waves by distributed inclusions
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Summary
We describe the 2-D elastodynamic boundary element method (BEM) in this paper and apply it to solve scattering problems. The method is based on the integral representation of a scattered wavefield by placing a fictitious source distribution on the surface of the scattering objects or inclusions. The fictitious sources can be determined by matching the appropriate boundary conditions at the bounding surfaces of the inclusions. This method, known as an indirect BEM, has the capability to calculate full wavefields including multiple scattering. The accuracy of the method is assured by comparing the BEM results with those calculated using other methods. We present some numerical examples of scattering of seismic waves by several different distributions of inclusions in order to demonstrate the versatility of the method.

Introduction
The scattering of elastic waves by cracks and other inhomogeneities is a long standing physical and mathematical problem, which together with the detection of voids inside a material in nondestructive evaluation, has long been of interest to geophysicists. Traditional approximate single scattering theories, such as the Born approximation based on the perturbation method (Hudson and Heritage, 1981; Wu, 1989) have serious limitations as they are only valid for weak elastic scattering. Another well-used method is the Kirchhoff approximation (e.g. Douglas and Hudson, 1990), which is valid only at high frequencies. For crack (diffraction) problems, the boundary element method (BEM) or boundary integral equation method is regarded as a natural choice because of its flexibility in defining the boundary of cracks. Like other numerical methods, such as finite difference and finite element methods, the BEM is a full wave modeling method and multi-scattering between cracks/pores can be included without additional difficulty.

Application of the BEM to wave propagation problems in complex media arises from the use of integral representations of the elastic wavefields or mathematical descriptions of Huygens' principle. The wavefield can be related to a certain boundary integral using a fictitious source distribution on each interface. The boundary integral can be transformed to a system of linear equations by discretizing the boundary into small elements, so that appropriate boundary conditions on crack surfaces can be applied. By solving these simultaneous linear equations, the fictitious source distributions can be obtained. The final step is to carry out an integral using the appropriate Green's function to obtain the total wavefield. In this paper, we aim to introduce one of the classes of BEMs, namely the indirect BEM, and apply it to solve the 2-D isotropic elastodynamic problems. Some numerical examples are presented to demonstrate the use of the BEM in dealing with scattering of seismic waves by various spatial distributions of inclusions.

Basic equations
In crack scattering problems the total wavefield is usually written as the superposition of the scattered field \( \mathbf{u}' \) and the free field \( \mathbf{u}^0 \) (i.e. the field in the absence of cracks):

\[
\mathbf{u} = \mathbf{u}^0 + \mathbf{u}'.
\]  

Consider the domain \( S \) of a crack or inclusion, and its boundary \( L \) (Figure 1): the scattered wavefield generated when a steady-state time-harmonic wave \( \mathbf{u}^0 \) is scattered by a void of arbitrary shape in an elastic solid, can be written as (Banerjee and Butterfield, 1981):

\[
\mathbf{u}'(\mathbf{x}) = \int_L \phi_i(\mathbf{x}') G_{ij}(\mathbf{x}, \mathbf{x}') dL', \quad i = 1, 2, 3,
\]

where \( u'_i \) is the \( i \)th displacement of scattered waves at \( \mathbf{x} \) and \( \phi \) is the fictitious source distribution evaluated at \( \mathbf{r} \) on \( L \) with outward normal \( \mathbf{n} \); \( G_{ij}(\mathbf{x}, \mathbf{x}') \) is Green's tensor: the displacement in the \( i \)th direction at the point \( \mathbf{x} \) due to the application of a unit force in the \( j \)th direction at the point \( \mathbf{x}' \).

The traction on \( L \) is given by (Banerjee and Butterfield, 1981):
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Figure 1. Problem configuration: A scattering object $S$ bounded by the curve $L$ with outward unit normal $n$.

\[ t_i(\mathbf{x}) = c_i \phi_i(\mathbf{x}) + \int_L \phi_j(\mathbf{x}') T_{ij}(\mathbf{x}, \mathbf{x}') dL', \quad i = 1, 2, 3, \]

(3)

where $T_{ij}$ is Green’s tensor for the traction; that is, the traction in the $i$th direction at the point $\mathbf{x}$ on the boundary due to the application of a unit force in the $j$th direction at the point $\mathbf{x}'$, and is related to the Green’s stress tensor by the Hooke’s law. The coefficient $c_i = 0.5$ when $\mathbf{x}$ approaches the boundary $L$ from inside $S$, and $c_i = -0.5$ when $\mathbf{x}$ tends to $L$ from outside (Pointer et al., 1998). The expression derived for the outward scattered field is also valid for a boundary $L$ made up of $N$ distinct boundaries $L^1, L^2, ..., L^n$ surrounding $n$ separate inclusions with domains $S^1, S^2, ..., S^n$. The scattered field expressed by equation (2) takes into account the interactions between inclusions, and we obtain the complete multi-scattered wavefield by:

\[ u_i(\mathbf{x}) = \sum_{n=1}^{N} \int_{L^n} \phi_j(\mathbf{x}') G_{ij}(\mathbf{x}, \mathbf{x}') dL'. \]

To evaluate the wavefield $u^e$ inside the inclusions, we treat the interior of each inclusion as an independent medium without reference to the exterior. By the same means as above we obtain expressions for $u^e$ and the corresponding traction in similar form to (2) and (3). Equations (2) and (3) are two boundary integrals governing the solution of any well-posed problem, and the boundary element method based on discretizing equations (2) and (3) is called the indirect BEM because the fictitious source distribution $\phi(\mathbf{x}')$ on $L$ has no physical meaning.

Discretization and BEM implementation

The essence of the BEM implementation is to discretize each boundary into a finite number of elements, and the appropriate boundary conditions, i.e. the continuity of displacement and stress across all elements, are then applied at each element. In the 2-D isotropic case, SH-waves are decoupled from P-SV waves, and can be treated separately.

For the simple antiplane case (SH-waves), the boundary conditions are the continuity of displacement $u_2$ and traction $t_3$ at each element $m$ on the boundary $L$ for the solid/solid contact. Putting the boundary conditions into equations (2) and (3) and after some algebra we can write the final results in condensed form:

\[ \sum_{m=1}^{M} \phi_m A_{nm} + \sum_{m=1}^{M} \phi'_m A'_{nm} = u_0^2, \]

(4)

and

\[ \sum_{m=1}^{M} \phi_m B_{nm} + \sum_{m=1}^{M} \phi'_m B'_{nm} = \ell_0^2, \]

(5)

for $n = 1, 2, ..., M$. The sub-matrices $A_{nm}, B_{nm}, A'_{nm}, B'_{nm}$ in equations (4-5) are related to the segment integral of the Green’s tensors. The terms on the right side of equations (4-5) are displacements and tractions of the incident waves at the surface $L$.

Similarly, in the case of P-SV waves, we have the following systems of linear equations using the boundary conditions for the continuity of displacements and tractions:

\[ \sum_{m=1}^{M} \phi_{jm} A_{ij}^{jm} + \sum_{m=1}^{M} \phi'_{jm} A'_{ij}^{jm} = u_0^i, \quad i = 1, 3, \]

(6)

and

\[ \sum_{m=1}^{M} \phi_{jm} B_{ij}^{jm} + \sum_{m=1}^{M} \phi'_{jm} B'_{ij}^{jm} = \ell_0^i, \quad i = 1, 3. \]

(7)
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for \( n = 1, 2, \ldots, M \). Coefficient sub-matrices \( A_{nm}, B_{nm}, A'_{nm}, B'_{nm} \) in equations (6-7) are related to the segment integral of the Green’s tensors. \( \phi_i \) and \( \phi'_i \) (\( i = 1, 3 \)) are the unknown fictitious surface stress distributions. The terms on the right side of equations (6-7) given are the displacements and tractions of incident waves on the boundary \( L \).

After solving these linear equations for \( \phi \) on the boundary \( L \), the final step is to compute displacements at any location \( x \) outside \( L \) through numerical integration of equation (2) for the \( i \)th component of displacement.

In general, equations (4-5) for SH-waves and (6-7) for P-SV waves form a system of \( 2M \) linear equations with \( 2M \) unknowns for the antiplane case (SH wave), and \( 4M \) for the inplane case (P-SV-waves) for a general solid/solid interface. This can be reduced if we consider some special cases, such as cracks filled with liquid or empty (gas-filled) cracks (Coutant, 1989). The coefficient matrices of the systems (4-5) and (6-7) are fully populated and are complex matrices and are non-symmetric. This is often regarded as the disadvantage of BEM in comparison with the finite element method. Nevertheless, this matrix can be easily manipulated as the number of the elements is not exceedingly high and the system of equations is only solved once for each frequency. In general, the number of elements depends on the highest frequency of interest; at low frequencies, a minimum number of element is required, while at high frequencies this number should be chosen such that at least three surface elements are sampled per seismic wavelength to give satisfactory results (Bouchon and Coutant, 1994).

Test of accuracy and numerical examples

In the first test example, the inclusion was assigned the same material properties as the surrounding medium. Clearly, this test should produce the free-field conditions, and in fact it did. The method was further validated by comparing our results with other results obtained using similar approaches. We compared the waveforms from scattering of SH-waves by a single cavity with the results obtained by Benítes et al. (1992) who used the boundary integral equation method. Mal (1970) computed the crack opening displacement for various wavelengths of excitation using a boundary integral equation method for a P-wave normally incident on a very thin crack located in an infinite elastic medium. In our model the crack contour is represented by 80 points (40 points for each face of the crack), and each end of the crack is represented with additional two points to make the crack tip more smooth. The distance between the two faces of the cracks is given a finite value which may be chosen arbitrarily small. The particular value used here is 1/1000 of the crack length. Our results are compared with Mal’s (1970) solution following Manolis and Besko (1988) in Figure 2. A good agreement has been achieved between our results and Mal’s (1970).

Figure 2. Amplitude of the displacement on the face of a crack calculated using Mal (1970) for a vertically incident P-wave for two frequencies. The result is normalized to the displacement at the centre of the crack. Curves are from Manolis and Besko (1988) and symbols are from BEM studied in this paper.

As a demonstration of the usefulness of the BEM, we present examples for the scattering of SH-waves by distributed cracks. Four different random realisations of the cracks are modeled (Figure 3). In each model, there are 30 cavities distributed in a 120mx120m area. Each crack has a half width of \( a = 2.5m \) and is discretized into 8 elements. The surrounding solid has \( v_p = 3500 \text{ m/s}, v_s = 2020 \text{ m/s}, \) and density \( \rho = 2.3 \text{ gcm}^{-3} \). A plane wave source is used that travels in the positive x-direction. 90 receivers are located parallel to the z-axis at \( x = 120 \text{ m} \) starting from \( z = 150 \text{ m} \) with an increment of \( \Delta z = 1.5 \text{ m} \). A Ricker wavelet with a dominant frequency of 100 Hz is used, so that \( k_p a = 0.45 \), and \( k_s a = 0.78 \) (\( k_p \) and \( k_s \) are P and S-wavenumbers), or \( \lambda_p/2a = 7 \) and \( \lambda_s/2a = 4 \) (\( \lambda_p \) and \( \lambda_s \) are P and S-wavelengths, respectively).
The resulting synthetic seismograms are given in Figure 4. The coda waves last longer for the SH-waves from models (c) and (d) and this can be explained by the fact that cracks are more clustered in the centre for models (a) and (b), whereas cracks are more scattered or more uniformly distributed for models (c) and (d). This is exactly what may be expected. A similar phenomenon can be seen for the corresponding P- and SV-wave wavefields (not shown here), but the wavefield is much more complicated than from the SH-wave sources, which is due to the inter-conversion between P- and SV-waves. A slight time delay can be seen in the middle of the plots (middle receivers) and this is due to the fact that the middle receivers are located immediately behind the cracks so that more scattering interferences are expected. These examples show that different distributions of inclusions have a significant influence on the multiple scattering and this may have an important implication in explaining coda waves in terms of randomly distributed crustal heterogeneities.

Conclusions

We have used a method based on boundary integral equations to solve scattering problems of elastic waves from inclusions. The method is known as the indirect BEM in which a fictitious surface source distribution over scattering objects can be found by matching the boundary conditions on the inclusion surfaces. The accuracy of the method has been tested and compared with other results. The BEM has the capability to calculate full wavefields including multiple scattering. We have computed multiple scattering wavefields in media containing cracks or inclusions with various random distributions to demonstrate the potential application of BEM in dealing with crack problems. Although we have only concentrated on a 2-D isotropic case, the method itself is applicable to full 3-D, and further extension to include general anisotropy is straightforward as long as a numerically simple Green's function can be found.

Acknowledgments

We thank David Booth for useful comments on the manuscript, and Colin MacBeth for useful discussion. This work was supported partly by Conoco (UK) Ltd and the Natural Environment Research Council, and is published with the approval of the Director of the British Geological Survey (NERC).

Reference


Manolis, G. D. and Besko, D. E., 1988, Boundary element methods in elastodynamics, Unwin Hy-
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Figure 4. Synthetic full wavefield from the plane SH-wave incidence. (a) to (d) correspond to crack distributions (a) to (d) in Figure 3. The numbers on the left side of the synthetic seismograms are the receiver numbers.