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Continuity conditions for a fault consisting of obliquely aligned cracks

S. R. Tod[1,2] and J. A. Hudson[1]

[1] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge,* CB3 9EW, *UK.*
E-mails: S.R.Tod@damtp.cam.ac.uk; J.A.Hudson@damtp.cam.ac.uk
[2] *British Geological Survey, West Mains Road, Edinburgh,* EH9 3LA, *UK*

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SUMMARY

Expressions exist for the continuity conditions that apply to a plane fault that is modelled as a distribution of circular cracks lying in the plane of the fault. These are conditions that apply when the wavelength of the incident wave is large compared with crack size and spacing, and they are approximate for a small number density of the cracks. In form they are identical to the 'fault-slip' conditions of Murty, Schoenberg and Pyrak-Nolte, Myer & Cook, in which the traction is proportional to the displacement discontinuity across the fault. It is possible to envisage a fault consisting of obliquely aligned cracks whose centres lie in the median plane of the fault but whose normals are not necessarily perpendicular to the fault. We have found the interface conditions for this model to be more complex than those for the fault-slip model and involve elements of the stress tensor other than the tractions on the plane of the fault. The continuity conditions are compared with those for a thin anisotropic layer and it is found that there is no equivalent material layer representation unless the cracks are parallel to the plane.

Key words: anisotropy, cracks, fault models.

1 INTRODUCTION

The effect of stress on geological materials is very complex. When loaded above the yield strength, a rock mass may deform in a continuous or plastic manner, it may slip along surfaces of weakness or pre-existing faults, and it may fail through the creation of new fractures. The pattern of fractures that is set up when a material fails for the first time can be quite varied, even in relatively homogeneous engineering materials (e.g. Naumann 1983). Faults in geological structures are generally assumed to be the results of tectonic shear stress of such a magnitude that fracture and slip occur across a more or less plane surface. The two sides of the fault no longer conform and the interface may be characterized by regions of firm contact, with the fault faces pressed together by the ambient stress, and other regions that are open and fluid- or air-filled (e.g. Nagy 1992). The fault may be seen as a distribution of cracks lying in a surface with the crack normals parallel to the normal of the surface. However, failure may be accompanied by cracks oriented in quite different directions (Naumann 1983). For instance, slip on a fracture can generate cracks striking away from the fracture surface. The plastic slip surfaces of a bar in torsion can be seen to form a criss-cross pattern on the surface of the bar, indicating that the surface of failure will lie at roughly 45° to the surface of maximum shear stress. Finally, materials with preferred planes of weakness will, in general, fracture along such planes. If the surface of maximum shear stress is inclined to these planes, the failure mechanism may show a sequence of short *en echelon* cracks. The cracks lie in the planes of weakness while their centroids lie in roughly the plane of shear, or fault plane.

Faults are significant and interesting features of geological structures. They may signify the orientation and magnitude of tectonic stresses over geological time. A fault can act as a fluid conduit or a seal depending on the fault zone complexity or microstructural details (Jones & Knipe 1996). Therefore, prediction of the fault seal potential is important. One way to characterize the fault zone complexities is to use seismic methods (Downey 1990; Yielding *et al.* 1992). In that they are structurally different from the surrounding rock, faults will have a seismic signature that, if it can be identified, would be a valuable tool for interpretation.

The simplest kind of fault, in which the fault surface is represented by a distribution of open regions interspersed with contact regions lying in the surface, has been modelled by several authors. The first seems to have been Sezawa & Kanai (1940), who proposed continuity of traction and normal displacement, and a discontinuity in transverse displacement proportional to the shear traction as a smoothed out representation of the motion at an incompletely welded interface of this kind. Similar empirical models by Murty (1976), Schoenberg (1980) and Pyrak-Nolte *et al.* (1990) have allowed for a general discontinuity in displacement linearly related to the traction, via a diagonal 'fracture compliance' matrix (the fault-slip model). A generalization of this model in which the fracture compliance matrix has non-zero off-diagonal terms has been proposed (Schoenberg & Douma 1988; Nakagawa *et al.* 2000), resulting in an 'anisotropic fault'. Such models have been shown to be applicable at long wavelengths for the fault model consisting of a distribution of cracks lying in a plane; they fail to take into account the scattering present at shorter wavelengths. Expressions have been derived for the coefficients in the relation between displacement discontinuity and traction in the case where the density of open regions, or cracks, is small (Hudson *et al.* 1996b) and where it is large (Hudson *et al.* 1997). The latter model has been used to interpret seismic data from vertical seismic profiles with good results (Worthington & Hudson 2000).

A straightforward extension of the model of the fault as a plane distribution of cracks lying in the fault plane (Hudson *et al.* 1996b) is one where the cracks are aligned with one another but with normals at an arbitrary angle to the fault plane (see Fig. 1). A 2-D numerical simulation of a periodic array of inclined cracks was carried out by Mikata & Achenbach (1988).

Following the discussion above, we may consider the fault plane to be the slip plane dictated by the stress field, while the cracks lie in planes of weakness of the material. For convenience we assume that the centroids of the cracks lie in the fault plane and, assuming that the cracks are of random shape, take the mean crack to be circular (see Fig. 2).

We analyse the effect that this crack distribution has on seismic waves using the method of smoothing and we work to second order in the number density of cracks. The results are valid for wavelengths long compared with the crack size and spacing, and for small crack number densities.

2 THEORY

Following Hudson *et al.* (1996b), the formula for the mean displacement $\langle \mathbf{u} \rangle$ in response to an incident field $\mathbf{u}^0$ scattered by a distribution of scatterers is

$$\langle \mathbf{u} \rangle = \mathbf{u}^0 + \varepsilon N \langle \mathbf{S}^1 \rangle \langle \mathbf{u} \rangle - \varepsilon^2 \left(N^2 \langle \mathbf{S}^1 \rangle^2 - N(N-1) \langle \mathbf{S}^1 \mathbf{S}^2 \rangle \right) \langle \mathbf{u} \rangle + \mathcal{O}(\varepsilon^3) , \quad (1)$$

where N is the total number of cracks, $\varepsilon \langle \mathbf{S}^1 \rangle$ is the mean scattering operator for the cracks and $\varepsilon^2 \langle \mathbf{S}^1 \mathbf{S}^2 \rangle$ is the mean of the operator for sequential scattering at two separate cracks.

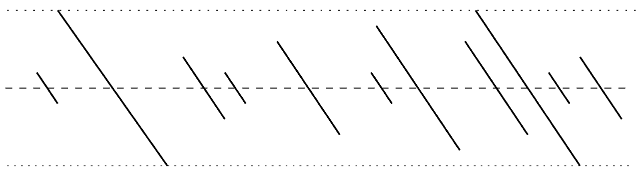

Figure 1. The fault model, consisting of a distribution of parallel cracks aligned at an angle to the fault plane (dashed line) in which they are centred and enclosed within the 'fault region' (dotted lines).

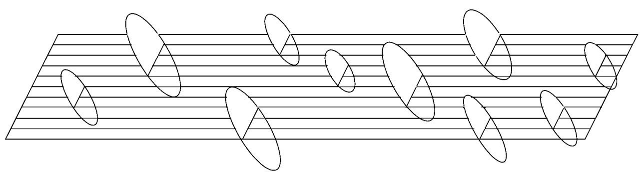

Figure 2. Schematic of a fault model in which the mean crack shape is taken to be circular.

We define the crack density ε to be

$$\varepsilon = \nu^s a^2 , \quad (2)$$

where ν^s is the number of cracks per unit area of fault and a is the mean diameter of a crack.

The mean scattering operator applied to the mean field is

$$\varepsilon N \langle S_i^1 \rangle \langle \mathbf{u} \rangle = \frac{a^3}{\mu} \int_{\mathscr{S}} \nu^s(\xi) \left[\frac{1}{a^2} \int_\Sigma U_{kl} \Big(t_l(\langle \mathbf{u} \rangle)|_\Sigma ; \xi + \mathbf{X} \Big) dS_\mathbf{X} \right] \times c^0_{kjmq} \frac{\partial G_i^m}{\partial \xi_q} (\mathbf{x}, \xi) n_j dS_\xi , \quad (3)$$

where $\mathbf{n}$ is the normal to the typical crack, $\mathbf{c}^0$ is the stiffness tensor of the homogeneous material in which the cracks are embedded,

$$c^0_{ipjq} = \lambda \delta_{ip} \delta_{jq} + \mu \left(\delta_{ij} \delta_{pq} + \delta_{iq} \delta_{pj} \right) , \quad (4)$$

$\mathscr{S}$ is the fault surface in which the centroids ξ of the cracks lie with number density $\nu^s(\xi)$, $\xi + \mathbf{X}$ is a point on the surface Σ of a typical crack, $\mathbf{t}(\langle \mathbf{u} \rangle)|_\Sigma$ is the traction on the crack due to the displacement field $\langle \mathbf{u} \rangle$, and $(a/\mu) U_{kl}(t_l; \xi + \mathbf{X})$ is the kth component of the discontinuity in displacement on the crack due to tractions $\mathbf{t}$ on the faces. The displacements $\mathbf{u}$ for any specific distribution of cracks are discontinuous across the face of each crack, but we expect the mean field, which is an expectation over all such distributions, to be continuous. However, we also expect that it will vary rapidly across the 'fault region', which we may define as a thin layer enclosing the cracks (see Fig. 1). If this is the case, the corresponding stresses will also vary continuously but rapidly across the fault region. The interface conditions that we derive are continuity conditions across the layer defining the fault region, in the case where the layer thickness is negligibly small compared with a wavelength.

To evaluate

$$\frac{1}{a^2} \int_\Sigma U_{kl} \Big(t_l(\langle \mathbf{u} \rangle)|_\Sigma ; \mathbf{X} \Big) dS_\mathbf{X} , \quad (5)$$

we may replace t_l by $\tilde{t}_l + t'_l$, the sum of a constant component, $\tilde{t}_l = \tilde{\sigma}_{lr} n_r$, where $\tilde{\sigma}_{lr}$ is the mean of the two values of $\sigma_{lr}(\langle \mathbf{u} \rangle)$ on either side of the fault region,

$$\tilde{\sigma}_{lr} = \frac{1}{2} \left[\sigma_{lr}|_{x_3 = 0_+} + \sigma_{lr}|_{x_3 = 0_-} \right] , \quad (6)$$

and t'_l is the remainder. Due to the linearity of U_{kl}, we may write eq. (5) as

$$\frac{1}{a^2} \int_\Sigma U_{kl} \Big(\tilde{t}_l|_\Sigma ; \mathbf{X} \Big) dS_\mathbf{X} + \frac{1}{a^2} \int_\Sigma U_{kl} \Big(t'_l|_\Sigma ; \mathbf{X} \Big) dS_\mathbf{X} . \quad (7)$$

Clearly t'_l changes sign across the crack face; if it were a completely anti-symmetric function, we would have

$$\frac{1}{a^2} \int_\Sigma U_{kl} \Big(t'_l|_\Sigma ; \mathbf{X} \Big) dS_\mathbf{X} = 0 , \quad (8)$$

and we assume this to be approximately true in all cases. Thus,

$$\varepsilon N \langle S_i^1 \rangle \langle \mathbf{u} \rangle = \frac{a^3}{\mu} \int_{\mathscr{S}} \nu^s(\xi) \bar{U}_{kl} \tilde{\sigma}_{lr}(\langle \mathbf{u} \rangle) n_r c^0_{kjmq} \frac{\partial G_i^m}{\partial \xi_q} (\mathbf{x}, \xi) n_j dS_\xi , \quad (9)$$

where

$$\bar{U}_{kl}=\frac{1}{a^2}\int_\Sigma U_{kl}(1;\mathbf{X})dS_\mathbf{X}\,, \tag{10}$$

and we have again used the fact that $U_{kl}(t_l;\mathbf{X})$ is linear in **t**.

Similarly,

$$\varepsilon^2\left(N^2\langle S_i^1\rangle\langle \mathbf{S}^2\rangle-N(N-1)\langle S_i^1\mathbf{S}^2\rangle\right)\langle\mathbf{u}\rangle$$
$$=\frac{a^3}{\mu}\int_{\mathscr{S}}\int_{\mathscr{S}} v^s(\boldsymbol{\zeta})[v^s(\boldsymbol{\xi})-v^s(\boldsymbol{\xi}|\boldsymbol{\zeta})]c^0_{kjmq}\,\frac{\partial G_i^m}{\partial\xi_q}\,(\mathbf{x},\,\boldsymbol{\xi})n_j\bar{U}_{kl}n_r$$
$$\times\, c^0_{lrpt}\,\frac{\partial}{\partial\xi_t}\left[A_{uvnx}\,\frac{\partial G_p^u}{\partial\zeta_v}\,(\boldsymbol{\xi},\,\boldsymbol{\zeta})\,\frac{\partial\widetilde{\langle u_n\rangle}}{\partial\zeta_x}\,(\boldsymbol{\zeta})\right]dS_\xi dS_\zeta\,, \tag{11}$$

where

$$A_{uvnx}=\frac{a^3}{\mu}\,c^0_{kjuv}c^0_{lrnx}n_jn_r\bar{U}_{kl} \tag{12}$$

and $v^s(\boldsymbol{\xi}|\boldsymbol{\zeta})$ is the conditional number density.

We wish to represent eqs (9) and (11) as radiation from discontinuities in displacement and traction on the fault $\mathscr{S}$. Suppose we have

$$\left.\begin{aligned}[\langle\mathbf{u}\rangle]&=\mathbf{A}(x_1,\,x_2)\\ [\langle\mathbf{t}\rangle]&=\mathbf{B}(x_1,\,x_2)\end{aligned}\right\}\ \text{on } x_3=0\,. \tag{13}$$

The associated radiation is given (e.g. Hudson 1980b) by **v**, where

$$v_i(\mathbf{x})=\int_{\mathscr{S}}\left[A_k(\xi_1,\,\xi_2)c^0_{k3pq}\,\frac{\partial G_i^p}{\partial\xi_q}\,(\mathbf{x},\boldsymbol{\xi})-B_k(\xi_1,\,\xi_2)G_i^k(\mathbf{x},\boldsymbol{\xi})\right]dS_\xi\,. \tag{14}$$

In order to identify this with the second and third terms in eq. (1), we need to bring it into the form

$$v_i(\mathbf{x})=\int_{\mathscr{S}} C_k(\xi_1,\,\xi_2)n_jc^0_{kjpq}\,\frac{\partial G_i^p}{\partial\xi_q}\,(\mathbf{x},\,\boldsymbol{\xi})dS_\xi\,. \tag{15}$$

To do this, we write

$$B_k(x_1,\,x_2)=\frac{\partial}{\partial x_1}\,B_{k1}(x_1,\,x_2)+\frac{\partial}{\partial x_2}\,B_{k2}(x_1,\,x_2)\,, \tag{16}$$

so that eq. (14) becomes

$$v_i(\mathbf{x})=\int_{\mathscr{S}}\left[A_k(\xi_1,\,\xi_2)c^0_{k3pq}\,\frac{\partial G_i^p}{\partial\xi_q}\,(\mathbf{x},\,\boldsymbol{\xi})\right.$$
$$\left.+B_{p\gamma}(\xi_1,\,\xi_2)\,\frac{\partial G_i^p}{\partial\xi_\gamma}\,(\mathbf{x},\,\boldsymbol{\xi})\right]dS_\xi\,, \tag{17}$$

where the suffix γ is summed over 1, 2 only.

For equality with eq. (15), we must have

$$\begin{aligned}&A_kc^0_{k3p\gamma}+B_{p\gamma}=C_kn_jc^0_{kjp\gamma}\,,\\ &A_kc^0_{k3p3}=C_kn_jc^0_{kjp3}\end{aligned} \tag{18}$$

for $\gamma=1$, 2, and $p=1$, 2, 3.

If the material outside the fault is isotropic, then

$$\begin{aligned}&A_\gamma=C_\gamma n_3+C_3n_\gamma\,,\\ &(\lambda+2\mu)A_3=\lambda(C_1n_1+C_2n_2)+(\lambda+2\mu)C_3n_3\,,\\ &(\lambda+2\mu)B_{11}=2\mu(2(\lambda+\mu)C_1n_1+\lambda C_2n_2)\,,\\ &B_{12}=\mu(C_1n_2+C_2n_1)=B_{21}\,,\\ &(\lambda+2\mu)B_{22}=2\mu(\lambda C_1n_1+2(\lambda+\mu)C_2n_2)\,,\\ &B_{3\gamma}=0\end{aligned} \tag{19}$$

for $\gamma=1$, 2. This corresponds exactly to the result derived for point dislocations by Hudson (1969).

Identifying eq. (15) with the sum of eqs (9) and (11), we have

$$C_k(\boldsymbol{\xi})=\frac{\varepsilon a}{\mu}\,\bar{U}_{kl}\left[\tilde{\sigma}_{lr}n_r-c^0_{lrst}n_rA_{uvnx}\int_{\mathscr{S}}[v^s-v^s(\boldsymbol{\xi}|\boldsymbol{\zeta})]\right.$$
$$\left.\times\,\frac{\partial^2G_s^u}{\partial\xi_t\partial\zeta_v}\,(\boldsymbol{\xi},\,\boldsymbol{\zeta})\,\frac{\partial\widetilde{\langle u_n\rangle}}{\partial\zeta_x}\,(\boldsymbol{\zeta})dS_\zeta\right]$$
$$=\frac{\varepsilon a}{\mu}\,\bar{U}_{kl}\left[\delta_{ln}+\frac{\varepsilon a}{\mu^2}\,c^0_{lqst}n_qc^0_{mjuv}n_j\bar{U}_{mn}K^s_{stuv}\right]\tilde{\sigma}_{nr}n_r\,, \tag{20}$$

where

$$K^s_{stuv}=-\mu\int_{\mathscr{S}}[1-n^s(\boldsymbol{\xi}-\boldsymbol{\zeta})]\,\frac{\partial^2G_s^u}{\partial\xi_t\partial\zeta_v}\,(\boldsymbol{\xi}-\boldsymbol{\zeta})dS_\zeta\,, \tag{21}$$

$$n^s(\boldsymbol{\xi}-\boldsymbol{\zeta})=\frac{v^s(\boldsymbol{\xi}|\boldsymbol{\zeta})}{v^s}\,, \tag{22}$$

and we have corrected a sign error in eq. (16) of Hudson *et al.* (1996b).

Now, if we let $\mathbf{X}=\boldsymbol{\xi}-\boldsymbol{\zeta}$, which represents the distance between two points on the fault plane $\mathscr{S}$, we have

$$K^s_{stuv}=\mu\int_{\mathscr{S}}[1-n^s(X)]\,\frac{\partial^2G_s^u}{\partial X_t\partial X_v}\,(\mathbf{X})dS_\mathbf{X}\,, \tag{23}$$

where $X=|\mathbf{X}|$, and we have further assumed that n^s depends on X only.

Exploiting the symmetries of eq. (20), we replace it by

$$C_k=\frac{\varepsilon a}{\mu}\,V_{kn}n_r\tilde{\sigma}_{nr}\,, \tag{24}$$

where

$$V_{kn}=\bar{U}_{kl}\left[\delta_{ln}+\frac{\varepsilon a}{\mu^2}\,c^0_{lqst}n_qc^0_{mjuv}n_j\bar{U}_{mn}\tilde{K}_{stuv}\right] \tag{25}$$

and

$$4\tilde{K}_{stuv}=K^s_{stuv}+K^s_{stvu}+K^s_{tsuv}+K^s_{tsvu}\,. \tag{26}$$

We may further symmetrize **C** by exploiting the symmetries of $\tilde{\boldsymbol{\sigma}}$, and thus replace $\tilde{\sigma}_{nr}$ in eq. (24) by (1/2) $(\delta_{np}\delta_{rq}+\delta_{rp}\delta_{nq})\tilde{\sigma}_{pq}$. We choose not to use this throughout the remainder of this paper for the sake of brevity.

We follow the same approach as Hudson (1980a) in approximating Green's function in the long-wavelength near-field limit, so the non-zero terms of the $\tilde{K}_{stuv}$ are

$$\tilde{K}_{\kappa\eta\gamma\tau} = \frac{1}{32}\int_0^\infty \frac{\partial n^s}{\partial X}\frac{dX}{X}\left[\left(1-\frac{\beta^2}{\alpha^2}\right)\delta_{\kappa\eta}\delta_{\gamma\tau} - \left(3+\frac{\beta^2}{\alpha^2}\right)(\delta_{\kappa\gamma}\delta_{\eta\tau}+\delta_{\kappa\tau}\delta_{\eta\gamma})\right],$$

$$\tilde{K}_{33\eta\tau} = \frac{1}{8}\int_0^\infty \frac{\partial n^s}{\partial X}\frac{dX}{X}\left(1-\frac{\beta^2}{\alpha^2}\right)\delta_{\eta\tau} = \tilde{K}_{\eta\tau 33},$$

$$\tilde{K}_{3\eta 3\tau} = \frac{1}{16}\int_0^\infty \frac{\partial n^s}{\partial X}\frac{dX}{X}\left(3-2\frac{\beta^2}{\alpha^2}\right)\delta_{\eta\tau} = \tilde{K}_{3\eta\tau 3} = \tilde{K}_{\eta 33\tau} = \tilde{K}_{\eta 3\tau 3},$$

$$\tilde{K}_{3333} = -\frac{1}{4}\int_0^\infty \frac{\partial n^s}{\partial X}\frac{dX}{X}\left(1-3\frac{\beta^2}{\alpha^2}\right) \tag{27}$$

for κ, η, γ, $\tau = 1, 2$.

Adopting the same functional dependence of n^s on X as Hudson *et al.* (1996b),

$$n^s(X) = \begin{cases} 1-e^{-(X-2a)^2/l^2}, & X \geq 2a \\ 0, & 0 \leq X < 2a \end{cases}, \tag{28}$$

where l represents the crack spacing, we find that

$$\int_0^\infty \frac{\partial n^s}{\partial X}\frac{dX}{X} = \pi(v^s)^{1/2}. \tag{29}$$

Thus eq. (1) may be seen to represent the mean field as a sum of the incident field together with radiation from discontinuities on the fault surface $\mathscr{S}$. The effect of the fault is, therefore, to introduce these discontinuities on $\mathscr{S}$; the discontinuity in displacement is **A**, which is given in terms of **C** by eq. (19), and the discontinuity in traction is **B**, which is given in terms of **C** by eqs (16) and (19); **C** is given by eqs (24) and (25).

Finally, to evaluate $\bar{U}_{kl}$ we use

$$\frac{a}{\mu}\bar{U}_{kl}(t_l) = [\langle u_k\rangle], \tag{30}$$

where $[\langle u_k\rangle]$ is the displacement discontinuity on a crack with normal **n**, under the influence of tractions **t** on the crack face.

We now change to new axes oriented with the 3-axis normal to the crack, i.e. along **n**. Let $\{l_{ij}\}$ be the rotation matrix, so that

$$l_{3i} = n_i, \quad l_{ji}n_i = \delta_{j3}. \tag{31}$$

The displacement referred to the new axes is

$$[\langle u_i'\rangle] = l_{ij}[\langle u_j\rangle] = \frac{a}{\mu} l_{ij}l_{pk}\bar{U}_{jk}(t_p') = \frac{a}{\mu}\bar{\mathscr{U}}_{ip}(t_p'), \tag{32}$$

where $\bar{\mathscr{U}}_{ip}(t_p')$ is μ/a times the discontinuity in displacement in the x_i-direction due to tractions $\mathbf{t}'$ acting on a crack with its normal in the x_3-direction. Expressions for $\bar{\mathscr{U}}_{ip}$ are given by Hudson (1980a, 1981, 1988) and Hudson *et al.* (1996b). Thus,

$$\bar{U}_{jk} = l_{pj}l_{qk}\bar{\mathscr{U}}_{pq}. \tag{33}$$

For circular cracks $\{\bar{\mathscr{U}}_{pq}\}$ is a diagonal matrix with $\bar{\mathscr{U}}_{11} = \bar{\mathscr{U}}_{22}$ (Hudson 1980a), so we have

$$\bar{U}_{jk} = (\delta_{jk} - n_jn_k)\bar{\mathscr{U}}_{11} + n_jn_k\bar{\mathscr{U}}_{33}. \tag{34}$$

Thus the continuity conditions for a fault consisting of obliquely aligned cracks are

$$[u_1] = \frac{\varepsilon a}{\mu}(n_3V_{1n}+n_1V_{3n})n_r\tilde{\sigma}_{nr},$$

$$[u_2] = \frac{\varepsilon a}{\mu}(n_3V_{2n}+n_2V_{3n})n_r\tilde{\sigma}_{nr},$$

$$[u_3] = \frac{\varepsilon a}{\mu}\left(\frac{\lambda}{\lambda+2\mu}(n_1V_{1n}+n_2V_{2n})+n_3V_{3n}\right)n_r\tilde{\sigma}_{nr},$$

$$[t_1] = \varepsilon a\left[2\left(\frac{2(\lambda+\mu)}{\lambda+2\mu}n_1V_{1n}+\frac{\lambda}{\lambda+2\mu}n_2V_{2n}\right)\frac{\partial}{\partial x_1} + (n_2V_{1n}+n_1V_{2n})\frac{\partial}{\partial x_2}\right]n_r\tilde{\sigma}_{nr},$$

$$[t_2] = \varepsilon a\left[2\left(\frac{2(\lambda+\mu)}{\lambda+2\mu}n_2V_{2n}+\frac{\lambda}{\lambda+2\mu}n_1V_{1n}\right)\frac{\partial}{\partial x_2} + (n_2V_{1n}+n_1V_{2n})\frac{\partial}{\partial x_1}\right]n_r\tilde{\sigma}_{nr},$$

$$[t_3] = 0. \tag{35}$$

We now examine the traction discontinuity, **[t]**. We first note that when the cracks are aligned in the plane of the fault, $n_1 = n_2 = 0$, **[t]** is identically zero and the continuity conditions reduce to those of Hudson *et al.* (1996b),

$$[\langle u_1\rangle] = \frac{\varepsilon a}{\mu}V_{11}\sigma_{13},$$

$$[\langle u_2\rangle] = \frac{\varepsilon a}{\mu}V_{11}\sigma_{23},$$

$$[\langle u_3\rangle] = \frac{\varepsilon a}{\mu}V_{33}\sigma_{33}, \tag{36}$$

with $\bar{U}_{kl} = \bar{\mathscr{U}}_{kl}$. It may be noted that since the traction is continuous across the fault, $\tilde{\boldsymbol{\sigma}}$ has been replaced by $\boldsymbol{\sigma}$ on the right-hand sides of these equations. We may also compare these with the empirical continuity conditions given by Pyrak-Nolte *et al.* (1990),

$$[u_1] = \frac{1}{\kappa_1}\sigma_{13},$$

$$[u_2] = \frac{1}{\kappa_2}\sigma_{23},$$

$$[u_3] = \frac{1}{\kappa_3}\sigma_{33}, \tag{37}$$

so that the two correspond exactly if

$$\kappa_1 = \kappa_2 = \left(\frac{\varepsilon a}{\mu}V_{11}\right)^{-1},$$

$$\kappa_3 = \left(\frac{\varepsilon a}{\mu}V_{33}\right)^{-1}. \tag{38}$$

A similar correspondence may be found with the empirical relations of Schoenberg (1980). Furthermore, under static loading perpendicular to the fault, or shear parallel to it, **[t]** vanishes, as it depends directly on derivatives parallel to the fault.

However, under dynamic conditions **[t]** is not zero in general and, as far as the authors are aware, no empirical conditions have been suggested that entail a discontinuity in traction, since such a discontinuity at an interface is normally taken to imply a distribution of uncompensated forces on the interface. The argument on which this is based relies on the process of taking a small disc of material, with one face above the interface and one below, and taking the limit as the thickness of the disc goes to zero. The disc then has zero mass but, if the tractions are unequal on either side of the interface, it is acted on by a finite force, thus implying a contradiction. However, in the case of the fault model we have chosen, this limit cannot be taken because the fault region is of finite, if small, thickness unless the cracks lie parallel to the fault, so there is, in fact, no contradiction in having a non-zero **[t]**.

The origin of the discontinuity in the traction is clear. Although the tractions are continuous across a crack, the stress tensor is not. Since the fault has a different orientation from that of the cracks, we should expect the tractions across the fault to be discontinuous.

If a plane wave of the form $\mathbf{q}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ is incident on the fault (whose normal is in the x_3-direction), all quantities will depend on x_1, x_2, t through the factor $e^{i(k_1x_1+k_2x_2-\omega t)}$. In this case the operators $\partial/\partial x_1$ and $\partial/\partial x_2$ in eq. (35) can be replaced, respectively, by ik_1 and ik_2.

3 CONTINUITY CONDITIONS FOR A THIN ANISOTROPIC LAYER

We compare our continuity conditions with those derived for a thin anisotropic layer of width h lying in $0 \le x_3 \le h$ by a finite difference approximation in the x_3-direction. We assume that the wave propagates in a medium whose properties vary only in the x_3-direction and we consider a solution to the equations of motion of the form

$$\mathbf{u} = \mathbf{F}(x_3)\, e^{i(k_1x_1+k_2x_2-\omega t)}\,; \tag{39}$$

this construction corresponds to a plane wave incident on the layer when it is set in homogeneous material above and below.

We integrate the equations of motion

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho' \frac{\partial^2 u_i}{\partial t^2} \tag{40}$$

and the stress–strain relations

$$e_{ij} = s'_{ijkl}\sigma_{kl} \tag{41}$$

across the layer, where $\mathbf{s}'$ is the tensor of compliances and ρ' is the density of the material in the layer. In doing so, we make the approximations

$$\begin{aligned} &\int_0^h \sigma_{kl}\,dx_3 \simeq h\tilde{\sigma}_{kl}, \\ &\int_0^h u_j\,dx_3 \simeq h\tilde{u}_j\,. \end{aligned} \tag{42}$$

In this way we obtain conditions for the discontinuities in displacements and tractions across the layer,

$$\begin{aligned} [u_1] &= 2hs'_{13kl}\tilde{\sigma}_{kl} - ik_1h\tilde{u}_3\,, \\ [u_2] &= 2hs'_{23kl}\tilde{\sigma}_{kl} - ik_2h\tilde{u}_3\,, \\ [u_3] &= hs'_{33kl}\tilde{\sigma}_{kl}\,, \\ [t_1] &= -h\rho'\omega^2\tilde{u}_1 - ih(k_1\tilde{\sigma}_{11} + k_2\tilde{\sigma}_{12})\,, \\ [t_2] &= -h\rho'\omega^2\tilde{u}_2 - ih(k_1\tilde{\sigma}_{12} + k_2\tilde{\sigma}_{22})\,, \\ [t_3] &= -h\rho'\omega^2\tilde{u}_3 - ih(k_1\tilde{\sigma}_{13} + k_2\tilde{\sigma}_{23})\,, \end{aligned} \tag{43}$$

and three further relations that do not contain terms involving discontinuities across the layer. Thus we find that this thin anisotropic layer exhibits a traction discontinuity; indeed, even for a thin isotropic layer we have such a discontinuity.

If the layer thickness is very much smaller than a wavelength, i.e.

$$kh \ll 1\,, \tag{44}$$

where k is the wavenumber of the incident wave, eq. (43) reduces to

$$\begin{aligned} [u_1] &= 2hs'_{13kl}\tilde{\sigma}_{kl}\,, \\ [u_2] &= 2hs'_{23kl}\tilde{\sigma}_{kl}\,, \\ [u_3] &= hs'_{33kl}\tilde{\sigma}_{kl}\,, \\ [t_1] &= -h\rho'\omega^2\tilde{u}_1\,, \\ [t_2] &= -h\rho'\omega^2\tilde{u}_2\,, \\ [t_3] &= -h\rho'\omega^2\tilde{u}_3\,. \end{aligned} \tag{45}$$

While boundary conditions for displacement discontinuities for thin anisotropic layers have been derived by many authors (e.g. Schoenberg & Douma 1988), the expressions obtained by Rokhlin & Huang (1993) also include a traction discontinuity of the same form as eq. (45).

The displacement and traction discontinuities are of the order

$$\begin{aligned} [\mathbf{u}] &\sim kh\,\frac{\|\mathbf{c}\|}{\|\mathbf{c}'\|}\,|\mathbf{u}|\,, \\ [\mathbf{t}] &\sim kh\,\frac{\rho'}{\rho}\,\|\boldsymbol{\sigma}\|\,, \end{aligned} \tag{46}$$

where $\mathbf{c}$ and ρ are the stiffness and density of the material outside the layer, thus we cannot neglect the displacement discontinuities if the material in the layer is much weaker than the matrix material; that is, $\|\mathbf{c}\|/\|\mathbf{c}'\|$ is sufficiently large that $kh\|\mathbf{c}\|/\|\mathbf{c}'\|$ is not negligibly small. This case was studied by Hudson (1981) and introduced by Hudson *et al.* (1996b) to construct an equivalent layer formulation for the continuity conditions (eq. 36) for a distribution of cracks lying in the fault plane.

On the other hand, the discontinuity in traction cannot be neglected if ρ'/ρ is sufficiently large that $kh\rho'/\rho$ is not small. This implies a very dense layer and, of course, it is physically very plausible that such a layer can be represented as a traction discontinuity.

The continuity conditions (eq. 45) for a thin anisotropic layer show some similarity with those for an array of parallel cracks (eq. 35) in that they involve discontinuities in traction, and the displacement discontinuities are related to all components of stress, not just the traction on the fault plane. However, the conditions for the parallel cracks include $[t_3]=0$, and this cannot be reproduced for a thin layer without having $[t_1]=0$ and $[t_2]=0$ as well. It appears, therefore, that there is no equivalent layer representation of a plane distribution of parallel cracks unless the cracks are parallel to the plane.

4 CONCLUSIONS

In generalizing the model of Hudson *et al.* (1996b), we have let the orientation of the cracks with respect to the fault plane—the plane in which the centroids of the cracks lie—vary, provided that they remain parallel to each other. This enables us to model a greater range of potential fault configurations.

We have derived the continuity conditions for this model and found that the traction is discontinuous across the fault. We also found that the discontinuities depend upon all elements of the average stress tensor, rather than just those components that constitute the traction on the fault. As far as the authors are aware, such conditions have not been postulated in any empirical models of an imperfectly bonded fault.

When the normal to the cracks is parallel to the normal to the fault, our conditions reduce to those of Hudson *et al.* (1996b), and are identical in form to empirical models (e.g. Murty 1976; Schoenberg 1980; Pyrak-Nolte *et al.* 1990).

Justification has been provided for the traction discontinuity, and we have further shown that traction is discontinuous across a thin layer of anisotropic material, even when the layer thickness is very much smaller than the wavelength, provided that the material in the layer is very much denser than the matrix material in which it is embedded. We have also shown that the displacement is discontinuous across the layer when the layer thickness is very much smaller than the wavelength, provided that the material in the layer is very much weaker than the matrix. However, it is clear that no material layer can be constructed that is equivalent, for wave propagation problems, to a distribution of parallel cracks unless the cracks are oriented with normals perpendicular to the fault plane. This does not prevent use of the boundary conditions (eq. 35) in reflection/transmission problems.

The model is independent of the nature of the crack infill, and we may account for different scenarios by varying the values of the parameters $\bar{\mathscr{U}}_{11}$ and $\bar{\mathscr{U}}_{33}$ to allow for fluid-filled cracks (Hudson 1980a), dry cracks or cracks filled with weak material (Hudson 1981), partially saturated cracks (Hudson 1988) and interconnected cracks (Hudson *et al.* 1996a). To allow for a number of sets of cracks with different orientations, we may calculate the contribution to $\bar{\mathscr{U}}_{ij}$ of each such set and form a weighted sum of these to calculate the total values of the $\bar{\mathscr{U}}_{ij}$ (Hudson 1986). To allow for a continuous distribution of crack orientation and aspect ratio, the sum becomes an integral in a straightforward way (see Hudson 1990; Peacock & Hudson 1990; Tod 2001).

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