NUMERICAL STUDY OF ELASTIC WAVE SCATTERING BY CRACKS OR INCLUSIONS USING THE BOUNDARY INTEGRAL EQUATION METHOD

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In this paper we use a 2D elastodynamic boundary integral equation or boundary element method (BEM) to solve multiple scattering problems due to existence of cracks or inclusions. The method is based on the integral representation of a scattered wavefield by assuming a fictitious source distribution on the scattering objects or inclusions (i.e. mathematical description of Huygens’ principle), and the fictitious source distribution can be found by matching appropriate boundary conditions at the boundary of the inclusions. The method is called as indirect boundary element method. Three numerical examples are presented to demonstrate the versatility of the BEM method. The first example shows that different spatial arrangements of the same scatters lead to profound differences in scattering characteristics, in particular the frequency contents of the transmitted wavefields using the method of time-frequency analysis. The second example shows the effects of power-law or fractal distribution of scalelengths on transmitted wavefields, and we conclude that frequency characteristics, such as the frequency of the peak attenuation, can be related to spatial size parameters of the model. In the third example, we show that orientated inclusions with aspect ratio less than unity have strong effects on the amplitudes of transmitted waves, and this has an important implication in characterizing inclusions and fractures using azimuthal variation in amplitudes (or attenuation anisotropy).

1. Introduction

When an elastic wave meets an object, it is scattered. If there are several objects (such as cracks or inclusions) the wavefield scattered from one object will induce further scattered

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fields from all the other objects, which will induce further scattered fields, from all the other objects, and so on. This process is called multiple scattering. The simplest approximation, called single scattering or the Born approximation, is to ignore the multiply scattered field completely. This approximation has been widely used (Wu, 1982; Hudson and Heritage, 1981). It is only valid for weak scattering or when the obstacles are small compared to both the wavelength and the spacing between the objects. Clearly, it has serious limitations when dealing with large scale inclusions or fractures such as in hydrocarbon reservoirs. Several theories exist for the computation of elastic wavefields which take multiple scattering into account, but few are valid for large crack sizes and short wavelengths except numerical approaches. When the size of inclusions is substantially less than wavelengths, various equivalent medium theories are available which produce azimuthal variation of elastic properties if there is a strong alignment of the inclusions (e.g. Hudson, 1981; Liu et al., 2000). The presence of spatial correlation of different crack systems cannot be accounted for with any effective medium theory. The use of numerical methods seems to be the only way which is capable of providing accurate solutions without restriction of size to wavelength ratio.

Finite difference methods (FDs) have been used widely in the study of scattering of elastic waves by crustal heterogeneities with continuous variation of physical properties (Frankel and Clayton, 1986), and they have also been used to model scattering by thin cracks (Fehler and Aki, 1978). However, it is not easy to handle discrete inhomogeneous bodies with FDs. In this paper, we use a method known as the 2D elastodynamic boundary integral equation or boundary element method (BEM) to compute wavefields from discrete inclusions with various spatial distributions. Note that previous applications of this method to elastodynamics include study of the effects of topography on wave propagation (Sánchez-Sesma and Campillo, 1991, 1993; Yokoi, 1996), wave propagation in laterally and smoothly varying media (Bouchon and Coutant, 1994), scattering of elastic waves by cavities (Bouchon, 1987; Murai et al., 1995; and Coutant, 1989); diffraction by hydraulic fractures (Liu et al., 1997; Pointer et al., 1998); and downhole source radiation (Dong and Tóksöz, 1995).

2. Integral representation of elastic wavefields

In scattering problems the total wavefield is usually written as the superposition of the scattered field $u^s$ and the free field $u^0$ (i.e. the field in the absence of any scatters):

$$u = u^0 + u^s. \quad (2.1)$$

Considering a domain $S$ surrounding a scattering object, and its boundary $L$ (Figure 1), the wavefield generated when a steady-state time-harmonic wave is scattered by a void of arbitrary shape in an elastic solid, can be derived using the reciprocal theorem (Cruse, 1968). In the indirect BEM representation, the scattered wavefield can be written as (Coutant, 1989; Sánchez-Sesma and Campillo, 1991; Pointer et al., 1998):

$$u^s_i(x) = \int_L \phi_j(x') G_{ij}(x,x') dL', \quad i = 1, 2, 3. \quad (2.2)$$
where $u^i_s$ is the $i$th component of displacement of scattered waves at $x$; and $\phi$ is the fictitious source evaluated at $x'$ on $L$ with outward normal $n$. Einstein summation convention is understood throughout. $G_{ij}(x, x')$ is the Green's tensor, i.e. the displacement in the $i$th direction at point $x$ due to the application of a unit force in the $j$th direction at point $x'$, and is given by (Pao and Varatharajulu, 1976):

$$G(x, x') = \frac{1}{4\pi \rho \omega^2} \left\{ k_s^2 I g_s(x, x') + \nabla \nabla \left[ g_p(x, x') - g_s(x, x') \right] \right\}, \quad (2.3)$$

where $I$ is unit matrix, and for the 2D wave propagation we have:

$$g_{p,s}(x, x') = i\pi H_0^{(1)}(k_{p,s} r), \quad (2.4)$$

where $\rho$ is density, $\omega$ is circular frequency, $k_p = \omega/v_p$, and $k_s = \omega/v_s$ are P- and S-wavenumbers, respectively ($v_p$ and $v_s$ are P- and S-wave velocities in solid); $r = |x - x'|$; and $H_0^{(1)}$ is the Hankel function of the first kind of order zero. Explicit Green's tensors can be found in Sánchez-Sesma and Campillo (1991); Pointer et al. (1998).

Fig. 1. Problem configuration: A scattering object $S$ bounded by the curve $L$ with outwards normal $n$. Upon an incidence of $u^0$ located at source the total wavefield received at receiver is the superposition of the incident wavefield $u^0$ and the scattered wavefield $u^s$.

The traction representation (Banerjee and Butterfield, 1981) is given by:

$$t_i(x) = c_i \phi_i(x) + \int_{L} \phi_j(x') T_{ij}(x, x') dL', \quad i = 1, 2, 3, \quad (2.5)$$

where $T_{ij}$ is Green's traction tensor, that is, the traction in the $i$th direction at point $x$ on the boundary due to the application of a unit force in the $j$th direction at point $x'$, and is
related to the Green’s stress tensor by Hook’s law. The coefficient \( c_1 = 0 \) when \( \mathbf{x} \) is not on the boundary \( L \); \( c_1 = 0.5 \) when \( \mathbf{x} \) approaches the boundary \( L \) from inside \( S \); and \( c_1 = -0.5 \) when \( \mathbf{x} \) tends to \( L \) from outside, provided the following two conditions are not violated (Banerjee and Butterfield, 1981; Pointer et al., 1998): (1) The point \( \mathbf{x}' \) is not located at any edge or a corner (i.e. there must be a unique tangent plane at \( \mathbf{x}' \)); and (2) The surface integral in Eq. (2.5) must be understood as a Cauchy principal-value integral.

The above equations deal with the presence of a single scatter. However, the expression derived for the outward scattered field is still valid for a boundary \( L \) made up of \( N \) distinct boundaries \( L_1, L_2, \ldots, L_N \) surrounding separate inclusions with surfaces \( S_1, S_2, \ldots, S_N \). The scattered field expressed by Eq. (2.2) takes into account the interactions between inclusions, and we obtain the complete multi-scattered wavefield:

\[
\mathbf{u}_i(\mathbf{x}) = \mathbf{u}_i^0(\mathbf{x}) + \sum_{n=1}^{N} \int_{L_n} \phi_i^n(\mathbf{x}')G_{ij}^n(\mathbf{x}, \mathbf{x}')dL'n. \tag{2.6}
\]

To achieve the single scattering approximation, we consider that each inclusion is only submitted to the incident wavefield but not to the scattered field radiated by other inclusions. Consequently, when computing the boundary conditions at a discretized point of a given surface, we must cancel in Eq. (2.6) each term describing the interaction between different inclusions; i.e. by setting the corresponding terms to zero before the inversion of the system.

To evaluate the wavefield inside the inclusions, we treat the interior of each inclusion as an independent medium with no interaction between other inclusions. Eqs. (2.2) and (2.5) are the two boundary integral equations governing the solution of any well-posed problem, and the boundary element method based on discretizing Eqs. (2.2) and (2.5) is called the indirect BEM because the fictitious source distribution \( \phi \) on \( L \) has no physical meaning.

### 3. Implementation of boundary element method

Eqs. (2.2) and (2.5) form the basis for the boundary element computation, but they are not useful unless coefficients \( \phi \) on the boundary \( L \) are known. To obtain \( \phi \), we also need to obtain an integral representation similar to Eqs. (2.2) and (2.5) for the interior material with an appropriate Green’s function (as in general, neither the displacement nor the stress vanishes on the diffraacting boundary). The essence of the BEM implementation is to discretize each boundary into a finite number of boundary elements, and the boundary conditions, i.e. the continuity of displacement and stress across all elements, are then applied at each element. In the 2D isotropic case, SH-waves are decoupled from P-SV waves, so we shall treat them separately.

#### 3.1. SH-wave case

In a simple antiplane case (SH-waves), the Eqs. (2.2) and (2.5) for \( i = 2 \) become:

\[
\mathbf{u}_2(\mathbf{x}) = \int_{L} \phi_2(\mathbf{x}')G_{22}(\mathbf{x}, \mathbf{x}')dL', \tag{3.1}
\]
Simulation of Elastic Wave Scattering by Inclusions

and

\[ t_2(x) = \frac{1}{2} \phi_2(x) + \int_L \phi_2(x') T_{22}(x, x') dL', \]  

(3.2)

and the choice of the sign in the first term of the right side of Eq. (3.2) depends on whether \( x \) is evaluated from inside or outside \( L \). The boundary conditions are the continuity of displacement \( u_2 \) and traction \( t_2 \) at each element on the boundary \( L \) for the solid/solid contact (assuming that the source is located outside the inclusions):

\[ u_2^0(x) + u_2^s(x) = u_2^i(x), \quad \text{and} \quad t_2^0(x) + t_2^s(x) = t_2^i(x), \quad \text{for} \ x \ on \ L, \]

(3.3)

where the terms with prime (\( ' \)) refer to the material of the crack interior. Displacements and stresses are calculated from the contribution of fictitious sources for all elements. These boundary conditions are satisfied at the centre of each element in the local co-ordinate system, which is defined such that the normal of each element is the positive axis and a clockwise right-hand co-ordinate system is assumed. In order to calculate the displacement at the \( n \)th element due to the source at the \( m \)th element, we discretize Eqs. (3.1) and (3.2) into \( M \) line segments \( \Delta L_m \) with normal \( \mathbf{n}_m \) (\( m = 1, 2, \ldots, M \)) using the Green’s function for both interior and exterior materials, and assume the force density per line unit is constant on each segment. We then have:

\[ u_2^0(x_n) + \sum_{m=1}^{M} \phi_2(x_m) \int_{\Delta L_m} G_{22}(x_n, x_m) dL' = \sum_{m=1}^{M} \phi_2(x_m) \int_{\Delta L_m} G_{22}(x_n, x_m) dL', \]

(3.4)

and

\[ t_2^0(x_n) - \frac{1}{2} \phi_2(x_n) + \sum_{m=1}^{M} \phi_2(x_m) \int_{\Delta L_m} T_{22}(x_n, x_m) dL' = \frac{1}{2} \phi_2(x_n) + \sum_{m=1}^{M} \phi_2(x_m) \int_{\Delta L_m} T_{22}(x_n, x_m) dL'. \]

(3.5)

The prime in the second variable in the Green’s functions is dropped without causing any confusion. The two integrals (3.4) and (3.5) contain, respectively, weakly- and strongly-singular kernels associated with displacement and traction Green’s tensors, and can be evaluated using the method given by Sánchez-Sesma and Campillo (1991) or Pointer et al. (1998). The above two equations can be re-written in the following condensed forms:

\[ \sum_{m=1}^{M} \phi_m A_{nm} + \sum_{m=1}^{M} \phi_m A'_{nm} = u_n^0, \]

(3.6)
and
\[ \sum_{m=1}^{M} \phi_m B_{nm} + \sum_{m=1}^{M} \phi'_m B'_{nm} = t^0_n, \]  
(3.7)

where we have written:
\[ \phi_m = \phi_2(x_m), \quad \text{and} \quad \phi'_m = \phi'_2(x_m), \]  
(3.8)

\[ u^0_n = -u^0_2(x_n), \quad \text{and} \quad \nu^0_n = -\nu^0_2(x_n). \]  
(3.9)

The terms on the right side of Eqs. (3.6) and (3.7) given in Eq. (3.9) are displacements and stress of the incident waves at the surface \( L \). The coefficient matrices \( A \)'s, \( B \)'s, \( A' \)'s and \( B' \)'s in Eqs. (3.6) and (3.7) are coefficient matrices and are given below:

\[ A_{nm} = G_{22}(x_n, x_m) = \int_{\Delta L_m} G_{22}(x_n, x_m) \, dl', \]  
(3.10)

\[ B_{nm} = -\frac{1}{2} \delta_{nm} + T_{22}(x_n, x_m) = -\frac{1}{2} \delta_{nm} + \int_{\Delta L_m} T_{22}(x_n, x_m) \, dl', \]  
(3.11)

\[ A'_{nm} = -G'_{22}(x_n, x_m) = -\int_{\Delta L_m} G'_{22}(x_n, x_m) \, dl', \]  
(3.12)

and

\[ B'_{nm} = -\frac{1}{2} \delta_{nm} - T'_{22}(x_n, x_m) = -\frac{1}{2} \delta_{nm} - \int_{\Delta L_m} T'_{22}(x_n, x_m) \, dl'. \]  
(3.13)

Eqs. (3.11) and (3.12) become

\[ B_{nm} = B'_{nm} = -\frac{1}{2}, \quad \text{if} \ n = m \]  
(3.14)

(Banerjee and Butterfield, 1984; Pointer et al., 1998). In all cases, \( n, m = 1, 2, ..., M \).

### 3.2. P-SV wave case

Similarly, in the in-plane case (or P-SV waves), we have the following boundary conditions for the continuity of normal and shear displacements:
\[ u_i^0(\boldsymbol{x}) + u_i^s(\boldsymbol{x}) = u_i^s(\boldsymbol{x}), \quad i = 1, 3, \quad \text{for } \boldsymbol{x} \text{ on } L, \]  
\hspace{1cm} (3.15)

and for the continuity of normal and shear stresses:

\[ t_i^0(\boldsymbol{x}) + t_i^s(\boldsymbol{x}) = t_i^s(\boldsymbol{x}), \quad i = 1, 3, \quad \text{for } \boldsymbol{x} \text{ on } L. \]  
\hspace{1cm} (3.16)

Putting Eqs. (2.2) and (2.5) into Eqs. (3.15) and (3.16) and following the same procedure as outlined for the case of the SH-wave incidence above, we can derive a system of linear equations similar to Eqs. (3.6) and (3.7) using the Green’s functions for both interior and exterior materials. The detailed derivation is omitted here, and only the final result is given in a condensed matrix form:

\[ \sum_{m=1}^{M} \phi_{jm} A^ij_{nm} + \sum_{m=1}^{M} \phi'_{jm} A^ij_{nm} = u^0_{im}, \quad i = 1, 3, \]  
\hspace{1cm} (3.17)

and

\[ \sum_{m=1}^{M} \phi_{jm} B^ij_{nm} + \sum_{m=1}^{M} \phi'_{jm} B^ij_{nm} = t^0_{im}, \quad i = 1, 3, \]  
\hspace{1cm} (3.18)

where again we have written:

\[ \phi_{im} = \phi_i(\boldsymbol{x}_m), \quad \text{and} \quad \phi'_{im} = \phi'_i(\boldsymbol{x}_m), \quad i = 1, 3, \]  
\hspace{1cm} (3.19)

\[ u^0_{im} = -u^0_i(\boldsymbol{x}_m), \quad \text{and} \quad t^0_{im} = -t^0_i(\boldsymbol{x}_m). \]  
\hspace{1cm} (3.20)

\( \phi_i \) and \( \phi'_i \) \((i = 1, 3)\) are the unknown fictitious surface stress distributions. The terms on the right side of Eqs. (3.17) and (3.18) given in Eq. (3.20) are the displacements and stress of incident waves on the boundary \( L \). The coefficient matrices \( A's, B's, A'i's, B'i's \) in Eqs. (3.17) and (3.18) are given below:

\[ A^ij_{nm} = \mathcal{T}_{ij}(\boldsymbol{x}_n, \boldsymbol{x}_m) = \int_{\Delta L_m} G_{ij}(\boldsymbol{x}_n, \boldsymbol{x}_m) \, dL', \]  
\hspace{1cm} (3.21)

\[ B^ij_{nm} = -\frac{1}{2} \delta_{nm} \delta_{ij} + \mathcal{T}_{ij}(\boldsymbol{x}_n, \boldsymbol{x}_m) = -\frac{1}{2} \delta_{nm} \delta_{ij} + \int_{\Delta L_m} T_{ij}(\boldsymbol{x}_n, \boldsymbol{x}_m) \, dL', \]  
\hspace{1cm} (3.22)
\[ A^{ij}_{nm} = -\overline{T}^{ij}_{nm}(\mathbf{x}_n, \mathbf{x}_m) = - \int_{\Delta L_m} G^{ij}_{nm}(\mathbf{x}_n, \mathbf{x}_m) \, dL', \quad (3.23) \]

and

\[ B^{ij}_{nm} = \frac{1}{2} \delta_{nm} \delta_{ij} - \overline{T}^{ij}_{ij}(\mathbf{x}_n, \mathbf{x}_m) = - \frac{1}{2} \delta_{nm} \delta_{ij} - \int_{\Delta L_m} T^{ij}_{ij}(\mathbf{x}_n, \mathbf{x}_m) \, dL'. \quad (3.24) \]

Eqs. (3.22) and (3.24) become

\[ B^{ij}_{nm} = B^{ij}_{nm} = \frac{1}{2} \delta_{ij}, \quad \text{if } n = m \quad (3.25) \]

(Banerjee and Butterfield, 1984; Pointer et al., 1998). In all cases, \( i, j = 1, 3 \), and \( n, m = 1, 2, \ldots, M \).

After solving these linear equations for \( \phi \) on the boundary \( L \), the final step is to compute displacements at any location \( \mathbf{x} \) outside \( L \) through numerical integration of the following formulae for the \( i \)th component of displacement [from Eqs. (2.1) and (2.2)]:

\[ u_i(\mathbf{x}) = u_i^{0}(\mathbf{x}) + \int_{L} \phi_j(\mathbf{x}') G_{ij}(\mathbf{x}, \mathbf{x}') \, dL', \quad i = 1, 2, 3. \quad (3.26) \]

Eq. (3.26) also needs to be discretized:

\[ u_i(\mathbf{x}) = u_i^{0}(\mathbf{x}) + \sum_{m=1}^{M} \phi_j(\mathbf{x}_m') \overline{G}_{ij}(\mathbf{x}, \mathbf{x}_m'), \quad i = 1, 2, 3, \quad (3.27) \]

where the elements of \( \{ \overline{G}_{ij} \} \) are given in Eqs. (3.10) and (3.21).

In general, Eqs. (3.6) and (3.7) for SH-waves and (3.17) and (3.18) for P-SV waves form a system of 2M linear equations with 2M unknowns for the antiplane case (SH-wave), and 4M for the inplane case (P-SV-waves) for a general solid/solid interface, and can be further reduced if we consider some special cases, such as inclusions filled with liquid or empty (gas-filled) inclusions. For the non-viscous liquid case, the boundary conditions are the continuity of normal displacements and normal stress, and vanishing of shear stress. For the empty crack case, boundary conditions are the vanishing of both normal and shear stress (stress-free boundary conditions). In both cases S-wave displacement is meaningless. For a detailed discussion, readers are directed to Coutant (1989) and Pointer et al. (1998).

The coefficient matrices of the equation systems (3.6) and (3.7) and (3.17) and (3.18) are fully populated complex matrices and are non-symmetric. This is often regarded as the disadvantage of BEM in comparison with finite element methods. Nevertheless, this matrix can be easily manipulated as the number of elements is not exceedingly high and...
the system of equations is only solved once for each frequency. A standard method such as the Gaussian elimination or $LU$ decomposition method can be used, and for large $M$, a conjugate gradient method can be used. For examples presented in this paper, the number of boundary elements is not too large, therefore we only use a standard $LU$ decomposition method to solve the linear equations. The maximum number of elements is restricted by the power of current computers and it also depends on the specified accuracy. In general, the number of elements depends on the particular frequency considered: at low frequencies, a minimum number of elements is required, while at high frequencies this number should be chosen such that at least three surface elements are sampled per wavelength to give satisfactory results (Bouchon and Coutant, 1994). However, the formulations given in this paper are valid for any $ka$ if the above condition is satisfied (where $k$ is wavenumber and $a$ is inclusion size).

![Graphs of Gaussian, Exponential, Uniform, and Gamma distributions with 2a=5m]{}

Fig. 2. Example 1: model used to compute synthetic seismograms from spatially distributed inclusions: (a) Gaussian, (b) exponential, (c) uniform; and (d) Gamma distributions. Plane waves (SH, SV and P) travel along the positive $x$-direction, and 90 receivers are located along the $z$-axis at $x = 120$ m starting from $z = 150$ m and with an increment of $\Delta z = -1.6$ m.

4. Numerical examples

Before we present numerical examples, it is necessary to mention that special care must be
taken for corners and edges as the constant $c_t$ given in Eq. (2.5) is not valid for non-smooth interfaces, such as corners or edges. There are two ways to circumvent this difficulty. The first and the simplest way is to take the field point to be slightly away from the corner by two separate nodes, i.e. by considering two corner nodes defined to be close to each other (typically 0.05 times the length of the local element apart). The second method is to re-calculate the constant $c_t$ using its original definition for a given corner or edge. Banerjee and Butterfield (1981) indicate that these two methods can give very similar results. In our implementation, we have used the first method.

Fig. 3. Synthetic full wavefield from the plane SH-wave incidence. (a) to (d) correspond to crack distributions (a) to (d) in Figure 2. The numbers on the left side of the synthetic seismograms are the receiver numbers.

The first example is given in Figure 2, which we attempt to model four different realisations of random crack distributions. In each model, there are 30 cavities (i.e. with stress-free boundary conditions) randomly distributed in a 120 m x 120 m area. Note that there is a problem of overlap of crack positions, and this was overcome automatically either by randomly moving adjacent inclusions so that the distance between the centre of adjacent inclusions is greater than the crack diameter, or by removing overlapping crack and generating another crack until the desired number of inclusions is reached. The second method is used in all examples in this paper. As a result of this process, the final crack distribution is not necessarily completely random, nevertheless, the purpose of this paper is to illustrate
the technique and to see how different distributions affect the multiple scattering. Each crack has a radius of \( a = 2.5 \) m, and is discretized into 8 elements. The crack surrounding solid (matrix) has \( v_p = 3500 \) m/s, \( v_s = 2020 \) m/s, and density \( \rho = 2.3 \) g/cm\(^3\). Plane wave sources are used. Plane waves travel along the positive \( x\)-direction, and 90 receivers are located along the line at \( x = 120 \) m starting from \( z = 150 \) m and with an increment of \( \Delta z = -1.6 \) m. A Ricker wavelet with a dominant frequency of 100 Hz is used, so that \( k_p a = 0.45 \) and \( k_s a = 0.78 \) (\( k_p \) and \( k_s \) are P- and S-wavenumbers), or equivalently \( \lambda_p / 2a = 7 \) and \( \lambda_s / 2a = 4 \) (\( \lambda_p \) and \( \lambda_s \) are P- and S-wavelengths, respectively).

The resulting synthetic seismograms are given in Figure 3 for SH-wave sources. [Note that the SH-waves are polarised out of plane in the \( y\)-direction, SV-waves in the \( z\)-direction and P-waves in the \( x\)-direction.] The coda waves last for a shorter period for the SH-waves in models (a) and (b) and this can be explained by the fact that inclusions are more clustered in the centre for models (a) and (b), whereas inclusions are more scattered or more uniformly distributed for models (c) and (d). A slight time delay can be seen in the middle of the plots (middle receivers) and this is due to the fact that the middle receivers are located immediately behind the crack clustering so that more scattering interferences are expected.

![Fig. 4. Comparison of synthetic seismograms and corresponding spectra from various distributions in Figure 3 (trace number 45).](image-url)

Figure 4 shows comparison of waveforms from the middle traces (number 45) of Figure 3 and their corresponding Fourier spectra. It is apparent that the amplitudes from uniform and Gamma distributions (traces c and d) are much smaller and have relatively low frequency contents. This presumably, is once again due to the fact that the cavities cluster together more closely in distributions (a) and (b) than in (c) and (d). In spite of the fact that the middle receiver lies directly behind the clustered cavities in (a) and (b), diffraction effects (at wavelength of around four cavity diameters) are able to maintain the amplitudes at this receiver by bringing energy around the sides of the cluster. With distributions (c) and (d), the spread of the cavities means that this reinforcement of the central trace is much reduced. The spectra for distributions (c) and (d) show a clear dip at 100 Hz, where the
cavity diameter is exactly a quarter wavelength and where a maximum in the scattering cross section might be expected to occur. This effect too is observed in (a) and (b) by diffraction. Further analysis using the time-frequency analysis shown in Figure 5 confirms not only the variation of frequency contents due to variation in crack position distributions, but also variation in duration of scattered or coda waves on the spatial variations. One of the advantages of the time-frequency analysis is that frequency contents from multiple arrivals, such as the scattered wavefields, can be identified. This example demonstrates that different distributions of inclusions have a significant influence on the multiple scattering.

The second example is used to model wave scattering from discrete inclusions with a scalelength distribution. The particular model that we use is given in Figure 6a, where with variation of crack sizes follows a von Kármán correlation function (Wu, 1982). Other correlation functions, such as Gaussian or exponential functions can also be easily calculated, and the use of von Kármán correlation function is purely for mathematical convenience. Readers are referred to the paper by Ikelle et al. (1993) for discussion about the generation of random media. The model given in Figure 6a is generated with a correlation length of 3.5 m and variance of also equal to 3.5 m. The largest crack radius is 5.5 m and smallest is 1.25 m. The peak frequency is 100Hz, which gives $ka$ ranging from 0.4 to 1.7 ($k$ is wavenumber and S-wave velocity is 2000 m/s). The source and receiver positions are arranged in the same way as described in the earlier example (Figure 3). Figure 6b shows the power spectrum of the crack size distributions shown in Figure 6a (plotted in log-log scale), and as expected, the variation can be fitted with a straight line. Such a model, i.e. with a linear variation of

![Graphs showing time-frequency analysis of synthetic traces](image-url)
Simulation of Elastic Wave Scattering by Inclusions

Fig. 6. (a) Example 2: model used to compute synthetic seismograms from crack distribution with power-law distribution of crack sizes. (b) Power spectra of crack size distributions as shown in (a). Horizontal axis is spatial wavenumber (in log scale).

Fig. 7. Synthetic acoustic wavefield corresponding to model shown Figure 6 (left plot), and comparison of scattering attenuation estimated from synthetic seismograms on the right side with by Wu’s (1982) single scattering solution.
power spectrum with spatial wavenumber, is often called fractal or power-law distribution (Leary, 1997). The corresponding synthetic SH-wavefield is given in Figure 7 (on the plot). As we can see, the wavefield is quite complicated, and there is long duration of coda wave energy. Figure 7 (on the right) shows the comparison of scattering attenuation estimated from the synthetic seismograms with single scattering solution derived by Wu (1982). The scattered attenuation is estimated using the approach described by Yomogida et al., (1997). We can see that there is a fairly good agreement between the BEM numerical results and single scattering solution of Wu (1982).

![Figure 7: The wavefield is quite complicated, and there is long duration of coda wave energy.](image)

**Fig. 7.** The wavefield is quite complicated, and there is long duration of coda wave energy. The corresponding synthetic SH-wavefield is given in Figure 7 (on the plot). As we can see, the wavefield is quite complicated, and there is long duration of coda wave energy. Figure 7 (on the right) shows the comparison of scattering attenuation estimated from the synthetic seismograms with single scattering solution derived by Wu (1982). The scattered attenuation is estimated using the approach described by Yomogida et al., (1997). We can see that there is a fairly good agreement between the BEM numerical results and single scattering solution of Wu (1982).

![Figure 8: Example 3: model used to compute synthetic seismograms from aligned inclusions with three different inclination angles from the propagation direction along the x-direction.](image)

**Fig. 8.** Example 3: model used to compute synthetic seismograms from aligned inclusions with three different inclination angles from the propagation direction along the x-direction.

![Figure 9: Synthetic P wavefield corresponding to model shown Figure 8. The letter 'R' on the top right corner denotes radial or x-component displacements and the letter 'V' denotes vertical or z-component displacements.](image)

**Fig. 9.** Synthetic P wavefield corresponding to model shown Figure 8. The letter 'R' on the top right corner denotes radial or x-component displacements and the letter 'V' denotes vertical or z-component displacements.

The last example is used to study the variation of transmitted P-wave amplitudes with azimuths in media with aligned elliptical inclusions. The model geometry is given in Figure 8, where the source and receivers geometry is exactly the same as model 1 shown in Figure 2. The synthetic seismograms computed for a source frequency of 100 Hz is given in Figure 8.
We can see the amplitudes computed for three azimuths vary significantly with azimuths in respect to the inclusion orientations, and the duration of coda wave energy depends on the orientations of aligned elliptical inclusions relative to the propagation direction, i.e. scattered wavefields have longer coda if inclusions are aligned in the direction which is not parallel to the propagation direction. Figure 10 shows the estimated attenuation ($Q^{-1}$) variations with azimuths. As expected, the attenuation or the amplitude decreases systematically from the direction parallel to the aligned inclusions. This example demonstrates that attenuation anisotropy may be used to provide useful information about the crack or inclusion alignment.

5. Conclusions

In this paper, we have shown that scattering by inclusions can be solved relatively easily using the boundary element method. Multiple scattering can be included without additional difficulty. The method has unique advantages over other numerical methods in a number of ways. Its primary disadvantage lies in the difficulty of modelling lateral variation, and high computer costs when (and only when) there are many inclusions to be discretized.

Numerical studies show that in the presence of inclusions, spatial and scalelength distributions, and orientations are important and cannot be ignored in modelling cracked rock. Different spatial arrangements of the same scatters lead to profound differences in scattering characteristics, in particular the frequency contents of the transmitted wavefields. The frequency characteristics, such as the frequency of the peak attenuation, and variation of attenuation with azimuths can be related to spatial size parameters and orientations of the
The complex characteristics of scattering wavefields from our examples have two immediate implications: On the negative side, increases in complexity due to multiple inclusions generate incoherent background scattered wavefields that tend to complicate the observations. So the interpretation becomes more difficult; On the positive side, this provides more insight into the complex mechanism of multiple scattering. By careful analysis, it should in principle be possible to obtain more information about the nature of spatially distributed inclusions.

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